Rend. Lincei Mat. Appl. (Online first)
DOI 10.4171/RLM/1051

Published by EMS Press

Algebraic Geometry. – *On the classification of product-quotient surfaces with* $q = 0$, $p_g = 3$ *and their canonical map*, by Federico Fallucca, communicated on 14 March 2025.

Abstract. – In this work, we present new results to produce an algorithm that returns, for any fixed pair of positive integers K^2 and χ, all regular surfaces S of general type with self-intersection of the canonical class $K_S^2 = K^2$ and Euler characteristic $\chi(\mathcal{O}_S) = \chi$, which are product-quotient surfaces. The key result we obtain is an algebraic characterization of all families of regular product-quotients surfaces, up to isomorphism, arising from a pair of G-coverings of $\mathbb{P}^1$. As a consequence of our work, we provide a classification of all regular product-quotient surfaces S of general type with $23 \leq K_S^2 \leq 32$ and $\chi(\mathcal{O}_S) = 4$. Furthermore, we study their canonical map and present several new examples of surfaces of general type with a high degree of the canonical map.

Keywords. – surfaces of general type, product-quotient, canonical maps.

Mathematics Subject Classification 2020. – 14J29.

Contents

Introduction

The history of the canonical map of surfaces of general type is more than 45 years long and it has been recently revived after the beautiful survey [27], where the authors provide an overview of the current state of knowledge on the topic, also outlining a series of still-open questions.

In 1978, Persson proved that the degree of the canonical map of surfaces of general type is bounded from above by 36, see [30]. Furthermore, it is known since [9] that if the degree is more than 27, then $q = 0$ and $p_g = 3$.

Surfaces of general type with a canonical map of degree d, where $3 \leq d \leq 9$, can be constructed quite easily as bi-double covers of a del Pezzo surface of degree d, see [27, Ex. 4.5]. However, constructing examples with a canonical map of higher degree, $d \geq 10$, becomes more challenging. For a long time, the only examples with a high degree of the canonical map were the surfaces of Persson [30] with degree 16 and Tan [34] with degree 12.

Recently, it has been proved that the bound given by Persson is sharp, see [26,32,35]. Recently, in [27], M. Mendes Lopes and R. Pardini revived the topic of the degree of the canonical map and posed in their survey, among other things, the natural question [27, Ques. 5.2] if all integers between 2 and 36 can be the degree of the canonical map of some surfaces of general type having $q = 0$ and $p_g = 3$.

It is also noteworthy, as mentioned in [9], that the degree of the canonical map is bounded from above by K_S^2, so that minimal surfaces with a high degree of the canonical map have not only $q = 0$ and $p_g = 3$ but also high values of K_S^2.

In this paper, we construct surfaces of general type with $q = 0$, $p_g = 3$, and $23 \leq K_S^2 \leq 32$ with the ultimate goal of determining the degree of their canonical map and providing new examples. These surfaces belong to the class of product-quotient surfaces.

Definition 0.1 ([4, Def. 0.1]). Let us consider a finite group G acting on two smooth projective curves C_1 and C_2, each of genus at least 2. We consider the diagonal action of G on $C_1 \times C_2$. Following [13, Rem. 3.10], we assume that the action on C_i is faithful.

If $X = (C_1 \times C_2)/G$ is smooth, which is equivalent to the action of G on the product $C_1 \times C_2$ being free, then we call X *product-quotient surface isogenous to a product*.

Otherwise, the minimal resolution of singularities S of $X = (C_1 \times C_2)/G$ is called *product-quotient* surface of the *quotient model* X.

We remind you that $K_S^2 = 32$ is the highest possible value for product-quotient surfaces with $q = 0$ and $p_g = 3$, see Theorem 2.4.

We consider product-quotient surfaces as they have proven to be highly useful tools in investigating unresolved conjectures in Algebraic Geometry. As a series of examples that only deal with regular surfaces, we mention the rigid but not infinitesimally rigid manifolds [6] constructed by Bauer and Pignatelli that gave a negative answer to a question of Morrow and Kodaira [28, p. 45], the classification of regular surfaces isogenous to a product of curves with $\chi(\mathcal{O}_S) = 2$ [23], the families of surfaces with $p_g = q = 0$ constructed in [4] realizing 13 new topological types and for which Bloch's conjecture [11] holds, and the series of papers [2,4,5,7,8] providing a classification of minimal product-quotient surfaces of general type with $p_g = q = 0$ to give a partial answer to a still-open problem posed by Mumford in 1980, see [3] and [8, p. 551].

As a first result of this paper, we refine the MAGMA [12] code of [4] and we present a new version of it which, taking as input a pair of positive integers K^2 and χ, returns all regular surfaces S of general type with self-intersection of the canonical class $K_S^2 = K^2$ and Euler characteristic $\chi(\mathcal{O}_S) = \chi$, which are product-quotient surfaces.

Although the original script is relatively easy to be adapted to any fixed value of χ and not only for $\chi = 1$ as in [4], it still presents computational time problems as the value of χ increases. We improve the code's efficiency by introducing new enhancements. To clarify these improvements, we briefly recall the algebraic description of regular product-quotient surfaces.

A regular product-quotient surface defines a pair of G-coverings of the projective line $C_i \to C_i/G \cong \mathbb{P}^1$, which can be algebraically characterized by finite sequences of elements of the group G satisfying certain conditions. These sequences are known as spherical systems of generators (cf. Definition 1.3). More precisely, any Galois covering of $\mathbb{P}^1$ can be associated with a finite group G, a set of (branch) points, and a spherical system of generators of the group G. Conversely, these data determine the Galois covering of $\mathbb{P}^1$. Thus, a regular product-quotient surface determines the following data:

- two sets of (branch) points in $\mathbb{P}^1$ and geometric loops around them;
- a finite group G;
- two spherical systems of generators of the group G.

Conversely, these data determine the product-quotient surface.

The geometry of a product-quotient surface can then be investigated by using the pair of spherical systems of generators associated with the corresponding pair of G-coverings of $\mathbb{P}^1$.

A first novelty of the code is the implementation of the database and the script *FindGenerators* developed in [15]. Such database contains one spherical system of generators of a finite group G for each family of pairwise topologically equivalent G-coverings C of $\mathbb{P}^1$, where the genus of C is $g(C) \leq 27$. We use these tools from [15] to speed up Step 3 in Section 2.1 as well.

The second main novelty is given from the following new result.

THEOREM 0.2. *Let V_1, V_2 be two spherical systems of generators of a finite group G. Assume that the associated topological types of G-coverings of $\mathbb{P}^1$ are different. The families of product-quotient surfaces associated with this pair of topological types of G-coverings are in natural bijection with the set of double cosets*

$$\mathcal{B}\mathrm{Aut}(G, V_1)\backslash \mathrm{Aut}(G)/\mathcal{B}\mathrm{Aut}(G, V_2).$$

This is a short version of the main Theorem 1.20. We also refer to the definitions in Section 1.2 that make clear the objects presented in Theorem 0.2. The analogous case of Theorem 0.2 where V_1 and V_2 have topological equivalent associated G-coverings of $\mathbb{P}^1$ is discussed in Corollary 1.22.

Techniques to establish whether two product-quotient surfaces belong to the same irreducible family have been extensively studied first in [7, Thm. 1.3] and [8, Prop. 5.2] in the case of surfaces isogenous to a product, and then in the general case in [4].

Theorem 1.20 seems to be a relevant new result on this problem, very useful in overcoming the huge amount of calculations that usually occur when adopting those techniques.

As a consequence of these improvements, we run the above-mentioned script to obtain a classification of regular product-quotient surfaces S with $23 \leq K_S^2 \leq 32$ and $\chi(\mathcal{O}_S) = 4$. What we obtain is the following.

THEOREM 0.3. *Let S be a regular product-quotient surface with $23 \leq K_S^2 \leq 32$ and $\chi(\mathcal{O}_S) = 4$. Then, S is a surface of general type and it realizes one of the families of surfaces described in Tables 9 to 21 in the appendix of this paper. Moreover, surfaces in Tables 9 to 20 are minimal.*

Apart from the rows of the tables where the number of families N is denoted by ?, the classification outlined in Theorem 0.3 yields a total of 1502 irreducible families of minimal surfaces of general type. Additionally, each family with $K^2 = 32$ maps onto an irreducible component (in the Zariski topology) of the Gieseker moduli space $\mathfrak{M}_{(4,32)}$, which consists of minimal surfaces of general type with $K_S^2 = 32$ and $\chi(\mathcal{O}_S) = 4$. The remaining cases, where $23 \leq K^2 \leq 30$, are more delicate and we refer to Section 1.2 and Remark 1.15.

We are interested in computing the degree of the canonical map of product-quotient surfaces, with a particular focus to those with $p_g = 3$.

Let S be a product-quotient surface given by a pair of curves C_1 and C_2 and a finite group G. We prove that the degree of the canonical map of S is determined whenever we compute the (schematic) base locus of the linear subsystem associated with the subspace $H^{2,0}(C_1 \times C_2)^G$ of invariant 2-forms of $C_1 \times C_2$.

Such subspace splits as a direct sum of subspaces $(H^{1,0}(C_1)^{\chi} \otimes H^{1,0}(C_2)^{\overline{\chi}})^G$, denoted for short by V_χ, one for each irreducible character $\chi \in \mathrm{Irr}(G)$. We need the following.

Property (#) A product-quotient surface S satisfies Property (#) if

$$\dim V_\chi \neq 0 \implies \deg(\chi) = 1$$

for each $\chi \in \mathrm{Irr}(G)$.

Remark 0.4. Property (#) always holds for G abelian group since each irreducible character of G has degree 1.

Assume that S satisfies Property (#). Then, Corollary 4.21 gives a formula for the base locus of each linear subsystem associated with the subspace V_χ, $\chi \in \mathrm{Irr}(G)$, and so of the base locus of $H^{2,0}(C_1 \times C_2)^G$ by intersecting them. Furthermore, Corollary 4.21 also implies that the canonical system $|K_S|$ of S is spanned by p_g divisors that are union of fibres (with multiplicity) for the natural fibrations $S \to C_i/G$, $i = 1, 2$.

In other words, Property (#) allows the degree of the canonical map of the product-quotient surfaces within a family to be computed, see Section 4.6 for an example. The degree is constant across the family and depends only on the pair of spherical systems of generators defining the family.

We also note that the formula for the degree of the canonical map is sharp in the sense that it cannot be improved by omitting Property (#). This is evidenced by examples such as no. 376 in Table 1, corresponding to those in [17], which describe regular product-quotient surfaces that violate this property. Despite sharing the same pair of spherical systems of generators, these surfaces have canonical maps of different degrees.

We have used the results obtained in Section 4 to produce a MAGMA code that computes the degree of the canonical map of a product-quotient surface with $q = 0$ and $p_g = 3$ satisfying Property (#).

We have then selected those surfaces in Theorem 0.3 satisfying Property (#) and we have computed the degree of their canonical map. We have obtained a series of examples that are listed in Table 1. The numbers of the column 'no.' of Table 1 refer to the row number of Tables 9 to 21 in the appendix. We refer to the appendix of this paper where we explain in detail all other information contained in the columns of Table 1. The examples of Table 1 with a high degree of the canonical map that are to our knowledge already discovered in the literature are the following:

- Surfaces of no. 42 of Table 1 are the examples presented in [18]. Other examples with a degree of the canonical map equal to 10 and 14 have been also constructed in [10] using a different approach.
- Families of surfaces no. 376 having a degree of the canonical map 12, (16, 18), (13, 15), 18 are all those in [17]. Furthermore, we point out that only surfaces no. 1 of [17, Thm. 2.3] satisfy Property (#) thanks to which the degree of their canonical map was automatically computable.

No.	K_S^2	Sing(X)	t_1	t_2	G	Id	N	deg(Φ_S)
1	32		2^6	2^8	$\mathbb{Z}_2^3$	$\langle 8,5\rangle$	3	$8, 16^2$
2	32		2^5	2^{12}	$\mathbb{Z}_2^3$	$\langle 8,5\rangle$	3	$0, 4, 8$
3	32		3^4	3^7	$\mathbb{Z}_3^2$	$\langle 9,2\rangle$	2	$6, 12$
4	32		3^5	3^5	$\mathbb{Z}_3^2$	$\langle 9,2\rangle$	1	9
5	32		$2^3, 4^2$	$2^3, 4^2$	$G(16,3)$	$\langle 16,3\rangle$	2	16
7	32		$2^2, 4^2$	$2^5, 4^2$	$G(16,3)$	$\langle 16,3\rangle$	6	8
9	32		$2^3, 4$	2^{12}	$\mathbb{Z}_2 \times D_4$	$\langle 16,11\rangle$	6	0
12	32		2^6	2^6	$\mathbb{Z}_2 \times D_4$	$\langle 16,11\rangle$	1	32
13	32		2^5	2^8	$\mathbb{Z}_2^4$	$\langle 16,14\rangle$	13	$0, 8^5, 16^7$
14	32		2^6	2^6	$\mathbb{Z}_2^4$	$\langle 16,14\rangle$	6	$8, 16^3, 32^2$
21	32		$2^2, 4^2$	$2^3, 4^2$	$G(32,22)$	$\langle 32,22\rangle$	7	16
28	32		2^5	2^6	$\mathbb{Z}_2^2 \times D_4$	$\langle 32,46\rangle$	4	24
42	32		7^3	7^3	$\mathbb{Z}_7^2$	$\langle 49,2\rangle$	7	$0, 5, 7, 10, 11, 14^2$
48	32		$2^2, 4^2$	$2^2, 4^2$	$\mathbb{Z}_2^5 \rtimes \mathbb{Z}_2$	$\langle 64,60\rangle$	3	32
87	30	$1/2^2$	$2^3, 4$	$2^{10}, 4$	$\mathbb{Z}_2 \times D_4$	$\langle 16,11\rangle$	6	0
88	30	$1/2^2$	$2^4, 4$	$2^5, 4$	$\mathbb{Z}_2 \times D_4$	$\langle 16,11\rangle$	2	4
119	28	$1/2^4$	$2^2, 4^2$	$2^8, 4^2$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\langle 8,2\rangle$	1	0
120	28	$1/2^4$	2^5	2^{11}	$\mathbb{Z}_2^3$	$\langle 8,5\rangle$	6	$0^2, 4^3, 8$
123	28	$1/2^4$	$2^3, 4$	2^{11}	$\mathbb{Z}_2 \times D_4$	$\langle 16,11\rangle$	14	0
124	28	$1/2^4$	2^5	$2^6, 4$	$\mathbb{Z}_2 \times D_4$	$\langle 16,11\rangle$	6	8
125	28	$1/2^4$	$2^2, 3^2$	$3^4, 6^2$	$\mathbb{Z}_3 \times S_3$	$\langle 18,3\rangle$	6	6^2
198	26	$1/2^6$	$2^3, 4$	$2^9, 4$	$\mathbb{Z}_2 \times D_4$	$\langle 16,11\rangle$	14	0
225	26	$1/3^2, 2/3^2$	$3, 9^2$	$3^2, 9^2$	$\mathbb{Z}_3 \times \mathbb{Z}_9$	$\langle 27,2\rangle$	6	$6^3, 7, 9, 10$
237	26	$1/3^2, 2/3^2$	$2, 6^2$	$2^4, 6^2$	$\mathbb{Z}_2^2 \times \mathcal{A}_4$	$\langle 48,49\rangle$	5	8
283	24	$1/2^8$	2^6	2^{10}	$\mathbb{Z}_2^2$	$\langle 4,2\rangle$	1	0
284	24	$1/2^8$	$2^3, 4^2$	$2^4, 4^2$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\langle 8,2\rangle$	1	8
285	24	$1/2^8$	$2^2, 4^2$	$2^7, 4^2$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\langle 8,2\rangle$	1	2
286	24	$1/2^8$	$2^2, 4^2$	$2^4, 4^4$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\langle 8,2\rangle$	2	$2, 8$
289	24	$1/2^8$	2^6	2^7	$\mathbb{Z}_2^3$	$\langle 8,5\rangle$	11	$4^3, 6^2, 8^3, 12^2, 16$
290	24	$1/2^8$	2^5	2^{10}	$\mathbb{Z}_2^3$	$\langle 8,5\rangle$	14	$0^4, 4^7, 6, 8^2$
295	24	$1/2^8$	$2, 4^3$	4^4	$\mathbb{Z}_4^2$	$\langle 16,2\rangle$	1	12
296	24	$1/2^8$	$2^2, 4^2$	$2^4, 4^2$	$\mathbb{Z}_2^2 \rtimes \mathbb{Z}_4$	$\langle 16,3\rangle$	13	8^3
298	24	$1/2^8$	$2^2, 4^2$	$2^4, 4^2$	$\mathbb{Z}_2^2 \times \mathbb{Z}_4$	$\langle 16,10\rangle$	10	$8^4, 12^4, 16^2$
303	24	$1/2^8$	$2^3, 4$	2^{10}	$\mathbb{Z}_2 \times D_4$	$\langle 16,11\rangle$	27	0
304	24	$1/2^8$	2^5	2^7	$\mathbb{Z}_2 \times D_4$	$\langle 16,11\rangle$	4	16
305	24	$1/2^8$	$2^4, 4$	2^6	$\mathbb{Z}_2 \times D_4$	$\langle 16,11\rangle$	14	8^2
308	24	$1/2^8$	2^5	2^7	$\mathbb{Z}_2^4$	$\langle 16,14\rangle$	13	$8^5, 12^4, 16^4$
309	24	$1/2^8$	$2^2, 3^2$	$3, 6^4$	$\mathbb{Z}_3 \times S_3$	$\langle 18,3\rangle$	3	$0, 6$
312	24	$1/2^8$	$2, 3^4, 6$	$2^2, 3^2$	$\mathbb{Z}_3 \times S_3$	$\langle 18,3\rangle$	3	6
376	24	$1/2^8$	$2, 3^2, 6$	$3, 6^2$	$S_3 \times \mathbb{Z}_3^2$	$\langle 54,12\rangle$	9	$12, (16,18), (13,15), 18, 24$
459	24	$1/4^2, 3/4^2$	$2^3, 4$	$2^9, 4$	$\mathbb{Z}_2 \times D_4$	$\langle 16,11\rangle$	6	0
475	23	$1/3^3, 2/3^3$	3^4	3^6	$\mathbb{Z}_3^2$	$\langle 9,2\rangle$	6	$6^5, 9$
477	23	$1/3^3, 2/3^3$	$2^2, 3^2$	$2^4, 3, 6$	$\mathbb{Z}_2 \times \mathcal{A}_4$	$\langle 24,13\rangle$	2	8
486	23	$1/3^3, 2/3^3$	$2, 6^2$	$2^4, 3, 6$	$\mathbb{Z}_2^2 \times \mathcal{A}_4$	$\langle 48,49\rangle$	6	8

TABLE 1. Product-quotient surfaces with $q = 0$, $p_g = 3$, and $23 \leq K^2 \leq 32$, whose canonical map degree has been computed.

- One of the families of no. 28 in Table 1 having 24 as degree of the canonical map has already been studied by the author of the present paper and D. Frapporti and can be found in [16, Sec. 6.3]. We also mention that this family of surfaces does not satisfy Property (#); hence, in this case, we have had to find the equations of the pair of curves realizing that family of surfaces and then studied by hand the degree of their canonical map. To our knowledge, there is only another example in the literature of a regular surface with a degree of the canonical map equal to 24 [31], which is constructed with a different technique.
- Two of the six families of no. 14 in Table 1 having degree 32 of the canonical map are discussed in [24]. They are described there differently from us, as $\mathbb{Z}_2^4$-coverings of $\mathbb{P}^1 \times \mathbb{P}^1$ using the language of Pardini's theory of abelian coverings [29]. Surfaces of these families are the only examples in the literature with a canonical map of degree 32, which is also the highest possible degree for product-quotient surfaces as observed in Remark 4.1.
  Furthermore, the authors proved in [24, Prop. 5.3] that these two examples are the only product-quotient surfaces with G abelian having degree of the canonical map equal to 32. The same question with G not abelian was still-open and it finds an answer in the present paper. Indeed, there are other families of surfaces in Table 1 with a canonical map of degree 32.

The paper is organized as follows.

In Section 1, we discuss finite group actions on a product of Riemann surfaces. We then present the main Theorem 1.20, the extended version of Theorem 0.2, crucial to speed up the classification algorithm for determining the number N of irreducible families.

In Section 2, we generalize [4, Prop. 1.14] to any $\chi \in \mathbb{N}$ and discuss the classification algorithm.

In Section 3, we prove Theorem 0.3. In particular, we show that all surfaces of Theorem 0.3 are of general type and those in Tables 9 to 20 are also minimal. We also discuss the exceptional cases arising from the secondary output of the function *ListGroups*$(K^2, 4)$ for each $K^2 \in \{23, \dots, 32\}$ in order to obtain the complete list of Tables 9 to 21 of the appendix.

In Section 4, we investigate the canonical map of product-quotient surfaces.

Section 5 is devoted to comparing the results obtained with our code to those in the literature, aiming at identifying any possible discrepancies.

An expanded version of Tables 9 to 21 of the appendix describing all the needed data to work explicitly with one of the surfaces and a commented version of the MAGMA codes we used can be found here.

Notation. We will use the basic notations of the theory of smooth complex projective surfaces; hence, K_S is the *canonical class* of S, $p_g := h^0(S, K_S)$ is the *geometric* *genus*, $q(S) := h^1(S, \mathcal{O}_S)$ is the *irregularity*, and $\chi(\mathcal{O}_S) = 1 - q + p_g$ is the *Euler characteristic*.

1. Algebraic characterization of families of product-quotient surfaces given by a pair of G-coverings of $\mathbb{P}^1$

Let S be a product-quotient surface of quotient model $(C_1 \times C_2)/G$. By a theorem due to Serrano [33, Prop. 2.2], $q(S) = 0$ if and only if $C_i/G \cong \mathbb{P}^1$.

In other words, pairs of G-coverings of the projective line define regular product-quotient surfaces. For this reason, let us briefly recall how coverings of $\mathbb{P}^1$ can be described.

1.1. *Algebraic characterization of families of G-coverings of $\mathbb{P}^1$*

Definition 1.1. Let G be a finite group. For a G-covering of $\mathbb{P}^1$ we mean a Riemann surface C together with a (holomorphic) action ϕ of G on C such that the quotient C/G is $\mathbb{P}^1$. Whenever we need to specify the action, we write (C, ϕ).

There are two notions of equivalence among G-coverings of $\mathbb{P}^1$: we say that C_1 and C_2 are *topologically equivalent* if there exists an orientation preserving homeomorphism $f : C_1 \to C_2$ and an automorphism $\varphi \in \mathrm{Aut}(G)$ such that $f(g \cdot p) = \varphi(g) \cdot f(p)$ for any $g \in G$ and $p \in C_1$. We say that C_1 and C_2 are *isomorphic* if moreover f is a biholomorphism.

Consider the set of G-coverings of $\mathbb{P}^1$ modulo isomorphism. The topological equivalence partitions it into equivalence classes, let $\mathcal{C}$ be one of them. González Díez and Harvey showed in [25] that $\mathcal{C}$ has a natural structure of connected complex manifold such that the natural map of $\mathcal{C}$ on the moduli space of curves mapping (C, ϕ) to C is analytic. More precisely, the manifold $\mathcal{C}$ is the normalization of its image $\widetilde{\mathcal{C}}$. In particular, $\widetilde{\mathcal{C}}$ is always an irreducible subvariety of the moduli space of curves.

The manifold $\mathcal{C}$ can be realized by taking a G-covering $C \in \mathcal{C}$ and moving the branch points of its covering map $C \to \mathbb{P}^1$, which endows C with a new holomorphic structure. Since the r branch points in $\mathbb{P}^1$ can be moved up to projective transformations, it follows that the dimension of the complex manifold $\mathcal{C}$ is $r - 3$.

Definition 1.2. We let $\mathcal{T}^r(G)$ be the collection of all classes of G-coverings of $\mathbb{P}^1$ ramified over r points modulo topological equivalence.

From the above discussion, we invite the reader to think of each element of $\mathcal{T}^r(G)$ as a class $\mathcal{C}$ of families of G-coverings of $\mathbb{P}^1$ pairwise not isomorphic but all topological equivalent to each other.

We shall give an algebraic description of the elements of $\mathcal{T}^r(G)$.

DEFINITION 1.3. A *spherical system of generators* (of length r) of G is a sequence of non-trivial elements $[g_1, \dots, g_r] \in G^r$ such that

$$G = \langle g_1, \dots, g_r \rangle, \quad \text{and} \quad g_1 \cdots g_r = 1.$$

The sequence $[o(g_1), \dots, o(g_r)]$ is called *signature* of $[g_1, \dots, g_r]$.

DEFINITION 1.4. We set $\mathcal{D}^r(G) \subset G^r$ to be the collection of all spherical systems of generators of G of length r.

REMARK 1.5. For each signature $[m_1, \dots, m_r]$ consider the *polygonal group*

$$\mathbb{T}(m_1, \dots, m_r) := \langle \gamma_1, \dots, \gamma_r | \gamma_1^{m_1}, \dots, \gamma_r^{m_r}, \gamma_1 \cdots \gamma_r \rangle.$$

There is a natural bijection between the set of *orbifold homomorphisms*, i.e. surjective homomorphisms $\varphi\colon \mathbb{T}(m_1, \dots, m_r) \to G$ such that any $\varphi(\gamma_i)$ has order m_i, and the set of spherical systems of generators of signature $[m_1, \dots, m_r]$.

The bijection associates with any homomorphism φ the spherical system of generators $[\varphi(\gamma_1), \dots, \varphi(\gamma_r)]$.

Consider the braid group $\mathcal{B}_r$, whose presentation with generators $\sigma_1, \dots, \sigma_{r-1}$ is

$$\mathcal{B}_r = \left\langle \sigma_1, \dots, \sigma_{r-1} \colon \begin{array}{rl} \sigma_i\sigma_j = \sigma_j\sigma_i, & |i-j| > 1 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j, & |i-j| = 1 \end{array} \right\rangle.$$

The group $\mathrm{Aut}(G) \times \mathcal{B}_r$ acts on $\mathcal{D}^r(G)$ as follows:

$$\begin{aligned} &\Psi \cdot [g_1, \dots, g_r] := \big[\Psi(g_1), \dots, \Psi(g_r)\big], \quad \Psi \in \mathrm{Aut}(G), \\ &\sigma_i \cdot [g_1, \dots, g_r] := [g_1, \dots, g_{i-1}, g_i \cdot g_{i+1} \cdot g_i^{-1}, g_i, g_{i+2}, \dots, g_r], \quad \sigma_i \in \mathcal{B}_r. \end{aligned}$$

The action of the generators σ_i extends to an action of the entire $\mathcal{B}_r$. These self-maps of $\mathcal{D}^r(G)$ are called *Hurwtiz moves*. We finally have the following classical result.

THEOREM 1.6. *The collection of all classes of G-coverings of $\mathbb{P}^1$ ramified over r points modulo topological equivalence is in bijection with $\mathcal{D}^r(G)/\,\mathrm{Aut}(G) \times \mathcal{B}_r$:*

$$\mathcal{T}^r(G) \cong \mathcal{D}^r(G)/\,\mathrm{Aut}(G) \times \mathcal{B}_r. \tag{1.1}$$

DEFINITION 1.7. A *topological type* of a G-covering of $\mathbb{P}^1$ is an element in $\mathcal{T}^r(G) \cong \mathcal{D}^r(G)/\,\mathrm{Aut}(G) \times \mathcal{B}_r$.

We briefly describe the bijection in Theorem 1.6 and refer to [22, Cor. 5.7] for a recent proof and further details on the topic. Consider an element in the quotient $\mathcal{D}^r(G)/\,\mathrm{Aut}(G) \times \mathcal{B}_r$, and choose a representative $[g_1, \dots, g_r]$. From Remark 1.5 we obtain an orbifold homomorphism $\mathbb{T}(m_1, \dots, m_r) \to G$, with $m_i := o(g_i)$. Next, we choose a finite set $X := \{q_1, \dots, q_r\}$ on $\mathbb{P}^1$, a base point $q_0 \in \mathbb{P}^1 \backslash X$, and a *geometric basis* of the fundamental group of $\mathbb{P}^1 \backslash X$. This basis consists of r distinct homotopy classes of loops η_i in $\mathbb{P}^1 \backslash X$, each starting at q_0 and traveling once around q_i counterclockwise, $i = 1, \dots, r$. These loops satisfy the relation $\eta_1 \cdots \eta_r = 1$, so $\mathbb{T}(m_1, \dots, m_r)$ is the quotient group of $\pi_1(\mathbb{P}^1 \backslash X, q_0)$ by the subgroup normally generated by $\eta_1^{m_1}, \dots, \eta_r^{m_r}$. The kernel of the composition

$$\pi_1(\mathbb{P}^1 \backslash X, q_0) \to \mathbb{T}(m_1, \dots, m_r) \to G$$

defines a unique topological G-covering of $\mathbb{P}^1 \backslash X$, which extends to a G-covering C of $\mathbb{P}^1$ by the Riemann Existence theorem.

The bijection of Theorem 1.6 maps the class of $[g_1, \dots, g_r]$ modulo $\mathrm{Aut}(G) \times \mathcal{B}_r$ to the class of C modulo topological equivalence.

Thus, C is a G-covering of $\mathbb{P}^1$ with branch points $q_1, \dots, q_r$, having ramification indices $m_1, \dots, m_r$ respectively, for which the Hurwitz formula holds:

$$2g(C) - 2 = |G| \left(-2 + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i} \right) \right). \tag{1.2}$$

Here, the cyclic groups $\langle g_i \rangle$ (and their conjugates) are the non-trivial stabilizers of the action of G on C. More precisely, g_i is the *local monodromy* of a point over q_i.

Definition 1.8. Let $q \in C' = C/G$ be a branch point of λ. The stabilizers of the points lying over q are cyclic subgroups of G and they are conjugated to each other. Thus, the order of the stabilizers depends only on q, denoted as m_q, the ramification index.

Let us fix a point $p \in \lambda^{-1}(q)$. Given a generator h of *Stab*(p), there exists a coordinate z in C such that the action of h in a neighborhood of p corresponds to $z \mapsto \delta z$, where δ is one of the m_q-roots of the unity. This gives a bijection among the primitive m_q-roots of the unity and the generators of *Stab*(p). We denote by *local monodromy* of p the unique generator of *Stab*(p) acting by $z \mapsto e^{\frac{2\pi i}{m_q}} z$.

Remark 1.9. The *local monodromy* of another point $g \cdot p$ over q is the conjugate ghg^{-1} of h. In other words, the *local monodromies* of points lying over q are conjugated to each other.

Let us give an example of how to use Theorem 1.6.

Example 1.10. We are going to compute $\mathcal{T}^3(S_3 \times \mathbb{Z}_3^2)$, the collection of the $S_3 \times \mathbb{Z}_3^2$-coverings of $\mathbb{P}^1$ up to topological equivalence ramified over 3 points. Up to apply suitable Hurwitz moves, we can assume that a spherical system of generators $[(g_1, v_1), (g_2, v_2), (g_3, v_3)]$ has $o(g_1) \le o(g_2) \le o(g_3)$. Observe $g_i \neq 1$; otherwise, S_3 would be generated by only one element, and this is not possible since it is not cyclic.

The same argument holds for $\mathbb{Z}_3^2$, so that $v_i \neq 0$. This implies $[g_1, g_2, g_3] \in \mathcal{D}^3(S_3)$, and $[v_1, v_2, v_3] \in \mathcal{D}^3(\mathbb{Z}_3^2)$. By Hurwitz formula (1.2), then $3\sum_{i=1}^3 \frac{1}{o(g_i)}$ has to be an integer, which holds only for either $o(g_1) = o(g_2) = o(g_3) = 3$ or $o(g_1) = o(g_2) = 2$, and $o(g_3) = 3$. The first case can be excluded since there are no g_1, g_2, g_3 of order 3 generating S_3.

Let us focus on the second case, which gives $g(C) = 0$, so $C \cong \mathbb{P}^1$. The elements of order 2 of S_3 are $\tau, \tau\sigma$, and $\tau\sigma^2$, where τ is a reflection (transposition) and σ is a rotation (3-cycle) of S_3. Since $g_3 = g_2^{-1}g_1^{-1}$, then $g_1 \neq g_2$; otherwise, $g_3 = 1$ since g_1 and g_2 have order two.

Thus, the list of spherical systems with ordered signature $[2, 2, 3]$ consists only of six elements obtained by choosing a distinct pair of g_1, g_2 in the set $\{\tau, \tau\sigma, \tau\sigma^2\}$. From here it is easy to see that the action of $\mathrm{Aut}(S_3)$ on $\mathcal{D}^3(S_3)$ is transitive.

On the other hand, it is clear that the action of $\mathrm{GL}_2(\mathbb{Z}_3)$ on $\mathcal{D}^3(\mathbb{Z}_3^2)$ is transitive. Thus, $\mathrm{Aut}(S_3 \times \mathbb{Z}_3^2)$ acts transitively on $\mathcal{D}^3(S_3 \times \mathbb{Z}_3^2)$, and from Theorem 1.6 we obtain

$$\mathcal{T}^3(S_3 \times \mathbb{Z}_3^2) \cong \frac{\mathcal{D}^3(S_3 \times \mathbb{Z}_3^2)}{\mathrm{Aut}(S_3 \times \mathbb{Z}_3^2) \times \mathcal{B}_3} = \{[(\tau, (1,0)), (\tau\sigma, (0,1)), (\sigma^2, (2,2))]\}.$$

By the Hurwitz formula (1.2), the genus of the corresponding G-covering C is $g(C) = 10$. Here, C may be described explicitly by equations as follows: we consider the projective space $\mathbb{P}^3$ with homogeneous coordinates $x_0, \ldots, x_3$ and define

$$C\colon \begin{cases} x_2^3 = x_0^3 - x_1^3, \\ x_3^3 = x_0^3 + x_1^3. \end{cases}$$

The action $\phi\colon S_3 \times \mathbb{Z}_3^2 \to \mathrm{Aut}(C)$ is given by

$$(\sigma^i\tau^j, (a,b)) \mapsto \left[(x_0 : x_1 : x_2 : x_3) \mapsto (\zeta_3^i x_{[j]} : x_{[j+1]} : (-1)^j \zeta_3^{2a+2i} x_2 : \zeta_3^{2b+2i} x_3)\right],$$

where $\zeta_3 := e^{\frac{2\pi i}{3}}$ is the first 3-root of the unity. Finally, the covering map by this action is

$$\lambda\colon C \xrightarrow{9:1} \mathbb{P}^1 \xrightarrow{6:1} \mathbb{P}^1, \quad (x_0 : x_1 : x_2 : x_3) \mapsto (x_0 : x_1) \mapsto (x_0^3 x_1^3 : (x_0^6 + x_1^6)/2).$$

REMARK 1.11. As we could expect, it becomes soon computationally difficult to get the $\mathrm{Aut}(G) \times \mathcal{B}_r$-orbits of $\mathcal{D}^r(G)$, as r or $|G|$ increases. For this reason, several authors put an increased effort into the development of an efficient algorithm to compute such orbits, usually with the help also of a computational algebra system (e.g. MAGMA, [12]). A big step forward in this direction is given for instance in [15], where the authors collect in a database a representative for each orbit of spherical systems of generators of fixed genus $g \leq 27$.

We use this database and their script *FindGenerators* to speed up Step 3 of Section 2.1 and in combination with Theorem 1.20 to give improvements in Step 5.

1.2. Families of product-quotient surfaces from a pair of coverings of $\mathbb{P}^1$

In this subsection, we study how to realize all families of product-quotient surfaces obtained by a pair of topological types of G-coverings of $\mathbb{P}^1$.

DEFINITION 1.12. Let us call by $\mathcal{T}^{r,s}(G)$ the collection of all families of regular product-quotient surfaces, whose associated (ordered pair of) G-coverings $\lambda_i\colon C_i \to \mathbb{P}^1$ are branched over r and s points, respectively.

REMARK 1.13. In the above definition, the order of C_1 and C_2 is relevant. Thus, exchanging them gives a natural bijection $\iota\colon \mathcal{T}^{r,s}(G) \to \mathcal{T}^{s,r}(G)$ which sends families to isomorphic families of surfaces.

In the previous section, we have seen that from a spherical system of generators in $\mathcal{D}^r(G)$ we can define an associated G-covering of $\mathbb{P}^1$, which realizes a family by moving the r branch points. Hence, a pair belonging to $\mathcal{D}^r(G) \times \mathcal{D}^s(G)$ gives a product-quotient surface, which realizes a family by moving respectively the r and s branch points of the attached G-coverings of $\mathbb{P}^1$. However, two pairs of spherical systems of generators in $\mathcal{D}^r(G) \times \mathcal{D}^s(G)$ may determine the same family of product-quotient surfaces; this occurs when they belong to the same orbit under the action of a certain group (see [2, 4] for more details).

PROPOSITION 1.14. *There is a natural bijection between* $\mathcal{T}^{r,s}(G)$ *and*

$$\frac{\mathcal{D}^r(G) \times \mathcal{D}^s(G)}{\mathrm{Aut}(G) \times \mathcal{B}_r \times \mathcal{B}_s},$$

where $\mathrm{Aut}(G)$ *acts simultaneously on both factors, whilst* $\mathcal{B}_r$ *and* $\mathcal{B}_s$ *act on the first and second factor, respectively.*

REMARK 1.15. We point out that each of the families of $\mathcal{T}^{r,s}(G)$ maps onto an algebraic subset of the Gieseker moduli space, but the images of two different families may not be distinct. This is because we are considering an equivalence relation among product-quotient surfaces which is weaker than the equivalence relation *being isomorphic*.

However, as proved in [8, Prop. 5.2], regular product-quotient surfaces S_1 and S_2 isogenous to a product are in the same irreducible component of the Gieseker moduli space if and only if their pair of spherical systems of generators share the same orbit by the action of $\mathrm{Aut}(G) \times \mathcal{B}_r \times \mathcal{B}_s$, possibly up to exchanging the factors.

To each family of product-quotient surfaces we have a naturally associated pair of topological types of G-coverings, thus giving a surjective map

$$\mathcal{T}^{r,s}(G) \twoheadrightarrow \mathcal{T}^r(G) \times \mathcal{T}^s(G).$$

By Proposition 1.14 and Theorem 1.6 we obtain the following commutative diagram:

$$\begin{tikzcd} \mathcal{T}^{r,s}(G) \arrow[r, leftrightarrow] \arrow[d, two heads] & \frac{\mathcal{D}^r(G)\times\mathcal{D}^s(G)}{\mathrm{Aut}(G)\times\mathcal{B}_r\times\mathcal{B}_s} \arrow[d, dashed, two heads, "\pi"] \\ \mathcal{T}^r(G)\times\mathcal{T}^s(G) \arrow[r, leftrightarrow] & \frac{\mathcal{D}^r(G)}{\mathrm{Aut}(G)\times\mathcal{B}_r}\times\frac{\mathcal{D}^s(G)}{\mathrm{Aut}(G)\times\mathcal{B}_s}. \end{tikzcd} \tag{1.3}$$

Here, π is defined as the unique map making the diagram commutative. Such π sends the class of a pair of spherical systems of generators $[V_1, V_2]$ to the pair of classes $([V_1], [V_2])$.

We are going to find the inverse image of each point $([V_1], [V_2])$ by π, which translates in determining each family of product-quotient surfaces afforded by the pair of topological types of G-coverings, the first given by $[V_1]$, and the second by $[V_2]$.

Definition 1.16. Let V be a spherical system of generators of length r of a finite group G. The group of automorphisms of *braid type* on V is the following subgroup of $\mathrm{Aut}(G)$:

$$\mathcal{B}\mathrm{Aut}(G, V) := \left\{\varphi \in \mathrm{Aut}(G) \colon \exists\, \sigma \in \mathcal{B}_r \text{ such that } \varphi \cdot V = \sigma \cdot V\right\}.$$

Since the action of an automorphism of G commutes with the action of a braid on a spherical system of generators, then it is immediate to see that $\mathcal{B}\mathrm{Aut}(G, V)$ is a subgroup of $\mathrm{Aut}(G)$: given $\varphi_1, \varphi_2 \in \mathcal{B}\mathrm{Aut}(G, V)$, then

$$(\varphi_1 \circ \varphi_2^{-1}) \cdot V = \varphi_1(\sigma_2^{-1} \cdot V) = \sigma_2^{-1} \cdot (\varphi_1 \cdot V) = (\sigma_2^{-1}\sigma_1) \cdot V$$

for some $\sigma_1, \sigma_2 \in \mathcal{B}_r$. Thus, $\varphi_1 \circ \varphi_2^{-1} \in \mathcal{B}\mathrm{Aut}(G, V)$.

Remark 1.17. If we replace V by V' in its $\mathrm{Aut}(G) \times \mathcal{B}_r$-orbit, let us say

$$V' := (\Psi, \sigma) \cdot V,$$

then the subgroup $\mathcal{B}\mathrm{Aut}(G, V')$ is conjugate to $\mathcal{B}\mathrm{Aut}(G, V)$:

$$\mathcal{B}\mathrm{Aut}(G, V') = \Psi \circ \mathcal{B}\mathrm{Aut}(G, V) \circ \Psi^{-1}.$$

Note that $\Psi \in \mathcal{B}\mathrm{Aut}(G, V)$ implies $\mathcal{B}\mathrm{Aut}(G, V') = \mathcal{B}\mathrm{Aut}(G, V)$.

Definition 1.18. Let V_1 and V_2 be a pair of spherical systems of generators of G. We will say that two automorphisms $\Phi, \Psi \in \mathrm{Aut}(G)$ are (V_1, V_2)-related, and we will write

$$\Phi \sim_{V_1,V_2} \Psi$$

if there exist $\varphi_1 \in \mathcal{B}\mathrm{Aut}(G, V_1)$, $\varphi_2 \in \mathcal{B}\mathrm{Aut}(G, V_2)$ such that

$$\Psi = \varphi_1 \circ \Phi \circ \varphi_2.$$

The relation $\sim_{V_1,V_2}$ is clearly an equivalence relation on $\mathrm{Aut}(G)$. We denote by $Q\mathrm{Aut}(G)_{V_1,V_2}$ the quotient of $\mathrm{Aut}(G)$ by $\sim_{V_1,V_2}$.

In other words, $\mathcal{Q}\mathrm{Aut}(G)_{V_1,V_2}$ is the set of double cosets

$$\mathcal{Q}\mathrm{Aut}(G)_{V_1,V_2} = \mathcal{B}\mathrm{Aut}(G,V_1)\backslash \mathrm{Aut}(G)/\mathcal{B}\mathrm{Aut}(G,V_2).$$

REMARK 1.19. Let V_1', V_2' be two spherical systems of generators belonging to the same orbits of V_1 and V_2, respectively, namely, $V_1' = (\Psi_1,\sigma_1)\cdot V_1$ and $V_2' = (\Psi_2,\sigma_2)\cdot V_2$. Then, by Remark 1.17, we have

$$\Phi \sim_{V_1,V_2} \Psi \iff \Psi_1\circ\Phi\circ\Psi_2^{-1} \sim_{V_1',V_2'} \Psi_1\circ\Psi\circ\Psi_2^{-1}.$$

Moreover, the bijection $\Phi \mapsto \Psi_1\circ\Phi\circ\Psi_2^{-1}$ induces a bijection among the quotients

$$\mathcal{Q}\mathrm{Aut}(G)_{V_1,V_2} \leftrightarrow \mathcal{Q}\mathrm{Aut}(G)_{V_1',V_2'}, \quad [\Phi]\mapsto[\Psi_1\circ\Phi\circ\Psi_2^{-1}] \tag{1.4}$$

which only depends on V_1, V_2, V_1', V_2' and not on the choice of Ψ_1, Ψ_2.

We can finally state and prove the main theorem of this section.

THEOREM 1.20. *Let π be the map $[V_1,V_2]\mapsto([V_1],[V_2])$ defined at (1.3). Let us fix a point*

$$x \in \frac{\mathcal{D}^r(G)}{\mathrm{Aut}(G)\times\mathcal{B}_r}\times\frac{\mathcal{D}^s(G)}{\mathrm{Aut}(G)\times\mathcal{B}_s},$$

and let us choose a pair of spherical systems of generators V_1 and V_2 such that $x = ([V_1],[V_2])$. The following hold:

(1) *Given $\Phi\in\mathrm{Aut}(G)$, then*

$$[V_1,\Phi\cdot V_2]\in\frac{\mathcal{D}^r(G)\times\mathcal{D}^s(G)}{\mathrm{Aut}(G)\times\mathcal{B}_r\times\mathcal{B}_s}$$

depends only on the class of Φ in $\mathcal{Q}\mathrm{Aut}(G)_{V_1,V_2}$.

(2) *The map*

$$\begin{aligned}\mathcal{Q}\mathrm{Aut}(G)_{V_1,V_2} &\to \pi^{-1}(x)\\ [\Phi] &\mapsto [V_1,\Phi\cdot V_2]\end{aligned}\tag{1.5}$$

is bijective. In particular, $|\pi^{-1}(x)| = |\mathcal{Q}\mathrm{Aut}(G)_{V_1,V_2}|$.

(3) *If we replace V_1 by V_1' in the same $\mathrm{Aut}(G)\times\mathcal{B}_r$-orbit, and V_2 by V_2' in the same $\mathrm{Aut}(G)\times\mathcal{B}_s$-orbit, then the bijective maps in (1.4) and (1.5) form a commutative triangle*

$$\begin{tikzcd} \mathcal{Q}\mathrm{Aut}(G)_{V_1',V_2'} \arrow[dd, leftrightarrow] \arrow[dr, leftrightarrow] & \\ & \pi^{-1}(x). \\ \mathcal{Q}\mathrm{Aut}(G)_{V_1,V_2} \arrow[ur, leftrightarrow] & \end{tikzcd}$$

Proof. (1) Let us consider an automorphism $\Phi' = \varphi_1 \circ \Phi \circ \varphi_2$ in the same class of Φ in $\mathcal{Q}\mathrm{Aut}(G)_{V_1,V_2}$, where $\varphi_1 \in \mathcal{B}\mathrm{Aut}(G, V_1)$ and $\varphi_2 \in \mathcal{B}\mathrm{Aut}(G, V_2)$:

$$\begin{aligned}
[V_1, \Phi' \cdot V_2] &= \left[V_1, (\varphi_1 \circ \Phi \circ \varphi_2) V_2\right] \\
&= \left[\varphi_1^{-1} \cdot V_1, (\Phi \circ \varphi_2) \cdot V_2\right] \\
&= \left[\sigma_1^{-1} \cdot V_1, \Phi \cdot (\sigma_2 \cdot V_2)\right] \\
&= \left[\sigma_1^{-1} \cdot V_1, \sigma_2 \cdot (\Phi \cdot V_2)\right] \\
&= [V_1, \Phi \cdot V_2].
\end{aligned}$$

(2) Point (1) proves that the map (1.5) is well defined. Let us consider an element $[V_1', V_2'] \in \pi^{-1}(x)$; hence, V_1' is in the same orbit of V_1 and V_2' is in the same orbit of V_2. We write

$$V_1' = (\Psi_1, \sigma_1) \cdot V_1 \quad \text{and} \quad V_2' = (\Psi_2, \sigma_2) \cdot V_2,$$

where $(\Psi_1, \sigma_1) \in \mathrm{Aut}(G) \times \mathcal{B}_r$, and $(\Psi_2, \sigma_2) \in \mathrm{Aut}(G) \times \mathcal{B}_s$. Then,

$$[V_1', V_2'] = [\Psi_1 \cdot V_1, \Psi_2 \cdot V_2] = \left[V_1, (\Psi_1^{-1} \circ \Psi_2) \cdot V_2\right].$$

This proves that the map (1.5) is surjective.

Let us consider $[\Phi_1]$ and $[\Phi_2]$ in $\mathcal{Q}\mathrm{Aut}(G)_{V_1,V_2}$ such that

$$[V_1, \Phi_2 \cdot V_2] = [V_1, \Phi_1 \cdot V_2].$$

We are going to show that $[\Phi_2] = [\Phi_1]$. Since $(V_1, \Phi_2 \cdot V_2)$ and $(V_1, \Phi_1 \cdot V_2)$ share the same orbit, then there exists $(\Psi, \sigma_1, \sigma_2) \in \mathrm{Aut}(G) \times \mathcal{B}_r \times \mathcal{B}_s$ such that

$$(V_1, \Phi_2 \cdot V_2) = (\Psi, \sigma_1, \sigma_2) \cdot (V_1, \Phi_1 \cdot V_2).$$

Then, we have

$$\Psi \cdot V_1 = \sigma_1^{-1} \cdot V_1 \quad \text{and} \quad (\Phi_1^{-1} \circ \Psi^{-1} \circ \Phi_2) \cdot V_2 = \sigma_2 \cdot V_2.$$

Therefore, defining $\varphi_1 := \Psi \in \mathcal{B}\mathrm{Aut}(G, V_1)$ and $\varphi_2 := \Phi_1^{-1} \circ \Psi^{-1} \circ \Phi_2 \in \mathcal{B}\mathrm{Aut}(G, V_2)$, we have

$$\Phi_2 = \varphi_1 \circ \Phi_1 \circ \varphi_2,$$

which proves $[\Phi_2] = [\Phi_1]$, and so that the map (1.5) is injective.

(3) It is an immediate consequence from the definition of the map (1.4). ■

Theorem 1.20 gives not only a perfect enumeration of the families of regular product-quotient surfaces corresponding to an *ordered* pair of topological types of G-coverings of the projective line (C_1, ϕ_1) and (C_2, ϕ_2) but also how to realize these families.

Indeed, given $\Psi \in \mathrm{Aut}(G)$, then (C_1, ϕ_1) and $(C_2, \phi_2 \circ \Psi^{-1})$ define an irreducible family of product-quotient surfaces. Theorem 1.20 translates as each family given by topological types of C_1 and C_2 is obtained in this way via an automorphism of $\mathrm{Aut}(G)$. Furthermore, two automorphisms Ψ_1 and Ψ_2 define the same family if they are (V_1, V_2)-related, or equivalently if their class in $\mathcal{Q}\mathrm{Aut}(G)_{V_1,V_2}$ is the same.

Thus, all families may be realized by a pair (C_1, ϕ_1) and $(C_2, \phi_2 \circ \Psi^{-1})$ via an automorphism representative Ψ for each class in $\mathcal{Q}\mathrm{Aut}(G)_{V_1,V_2}$.

We consider *ordered* pairs of topological types because of Remark 1.13, where we have observed that exchanging C_1 and C_2 defines an involution on $\bigcup \mathcal{T}^{r,s}(G)$ connecting isomorphic families.

If we are interested in counting the families given by two different topological types of G-coverings, then it is sufficient to choose an order on them and then apply Theorem 1.20.

However, to enumerate the families of product-quotient surfaces associated with twice the same topological type, we need to study how the exchange of the factors acts on $\mathcal{Q}\mathrm{Aut}(G)_{V,V}$.

PROPOSITION 1.21. *The exchange of the factors acts on $\mathcal{Q}\mathrm{Aut}(G)_{V,V}$ as the involution*

$$\mathcal{Q}\mathrm{Aut}(G)_{V,V} \to \mathcal{Q}\mathrm{Aut}(G)_{V,V}, \quad [\Phi] \mapsto [\Phi^{-1}].$$

PROOF. The exchange of the factors is a map from $\pi^{-1}([V],[V])$ to itself sending each $[V, \Phi \cdot V]$ to $[\Phi \cdot V, V] = [V, \Phi^{-1} \cdot V]$. ■

COROLLARY 1.22. *Let C_1 and C_2 be two G-coverings of $\mathbb{P}^1$ and let V_1 and V_2 be spherical systems of generators of them. Then, the cardinality of the set of families of product-quotient surfaces given by the topological types of C_1 and C_2 is equal to*

(1) *the cardinality of $\mathcal{Q}\mathrm{Aut}(G)_{V_1,V_2}$ if C_1 and C_2 are not topological equivalent;*
(2) *the cardinality of $\mathcal{Q}\mathrm{Aut}(G)_{V_1,V_1}/(\Phi \mapsto \Phi^{-1})$ if C_1 and C_2 are topological equivalent.*

Let us give an example how we use Theorem 1.20 and Corollary 1.22.

EXAMPLE 1.23. Let $G = S_3 \times \mathbb{Z}_3^2$. We are going to compute all regular product-quotient surfaces with quotient model $(C_1 \times C_2)/G$ where the G-coverings $\lambda_1 : C_1 \to \mathbb{P}^1$ and $\lambda_2 : C_2 \to \mathbb{P}^1$ are both ramifying over three points.

Keeping the notation of Example 1.10, we have seen there that C_1 and C_2 are described by

$$V := \big[\big(\tau, (1,0)\big), \big(\tau\sigma, (0,1)\big), \big(\sigma^2, (2,2)\big)\big].$$

We need to compute the subgroup $\mathcal{B}\mathrm{Aut}(G, V) \leq \mathrm{Aut}(S_3 \times \mathbb{Z}_3^2)$.

Firstly, we note that

$$\mathrm{Aut}(S_3 \times \mathbb{Z}_3^2) \cong \mathrm{Aut}(S_3) \times \mathrm{GL}_2(\mathbb{Z}_3).$$

Hence, every element of $\mathcal{B}\mathrm{Aut}(G, V)$ can be written as a pair (Ψ, M), where $\Psi \in \mathrm{Aut}(S_3)$, and $M \in \mathrm{GL}_2(\mathbb{Z}_3)$.

The action of $\mathcal{B}_3$ on $[(1, 0), (0, 1), (2, 2)]$ permutes its entries since $\mathbb{Z}_3^2$ is abelian. Therefore, the automorphisms $M \in \mathrm{GL}_2(\mathbb{Z}_3)$ of braid type on it are those permuting its entries. Such automorphisms belong to the subgroup $\langle M_1, M_2\rangle \cong S_3$ generated by

$$M_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad M_2 := \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}.$$

Let (Ψ, M) be of braid type on V, and let η be a braid in $\mathcal{B}_3$ such that $(\Psi, M) \cdot V = \eta \cdot V$. We observe that the signature of V is $[6, 6, 3]$: since the third number is different from the other two, and the automorphisms preserve the order, then the permutation image of η in S_3 fixes the number three. This implies that M fixes $(2, 2)$, so $M \in \langle M_1\rangle \cong \mathbb{Z}_2$. Therefore,

$$\mathcal{B}\mathrm{Aut}(G, V) \leq \mathrm{Aut}(S_3) \times \langle M_1\rangle \cong S_3 \times \mathbb{Z}_2.$$

Let us choose two generators of $\mathrm{Aut}(S_3)$: let Ψ_1 be the inner automorphism given by τ and let Ψ_2 be the inner automorphism induced by σ^2. We observe that (Ψ_1, Id) and (Ψ_2, M_1) are of braid type on V since they act on V, respectively, as the braids $\sigma_1\sigma_2^2\sigma_1$ and σ_1. Since they generate the whole $\mathrm{Aut}(S_3) \times \langle M_1\rangle$, then

$$\mathcal{B}\mathrm{Aut}(G, V) = \mathrm{Aut}(S_3) \times \langle M_1\rangle \cong S_3 \times \mathbb{Z}_2.$$

Now, we can compute $\mathcal{Q}\mathrm{Aut}(S_3 \times \mathbb{Z}_3^2)_{V,V}$, which as observed is the set of double cosets

$$\mathcal{Q}\mathrm{Aut}(S_3 \times \mathbb{Z}_3^2)_{V,V} = {}_{\mathcal{B}\mathrm{Aut}(G,V)}\backslash^{(\mathrm{Aut}(S_3)\times\mathrm{GL}_2(\mathbb{Z}_3))}/_{\mathcal{B}\mathrm{Aut}(G,V)}.$$

Since $\mathcal{B}\mathrm{Aut}(G, V) = \mathrm{Aut}(S_3) \times \langle M_1\rangle$ contains the subgroup $\mathrm{Aut}(S_3) \times \{1\}$, which is normal in $\mathrm{Aut}(S_3) \times \mathrm{GL}_2(\mathbb{Z}_3)$, then we have the following natural identification:

$$\mathcal{Q}\mathrm{Aut}(S_3 \times \mathbb{Z}_3^2)_{V,V} \cong {}_{\langle M_1\rangle}\backslash^{\mathrm{GL}_2(\mathbb{Z}_3)}/_{\langle M_1\rangle}. \tag{1.6}$$

More precisely, the correspondence sends $[(\mathrm{Id}_{S_3}, A)] \leftrightarrow [A]$.

From diagram (1.3) and Theorem 1.20, we can conclude that

$$\mathcal{T}^{3,3}(S_3 \times \mathbb{Z}_3^2) \cong \mathcal{Q}\mathrm{Aut}(G)_{V,V} \cong {}_{\langle M_1\rangle}\backslash^{\mathrm{GL}_2(\mathbb{Z}_3)}/_{\langle M_1\rangle}.$$

However, we are majorly interested in finding the set of families of product-quotient surfaces given by the pair (V, V). As proved in Corollary 1.22, it is sufficient to determine

$$\mathcal{Q}\mathrm{Aut}(G)_{V_1,V_1}/(\Phi \mapsto \Phi^{-1}).$$

This is the quotient of $\mathrm{GL}_2(\mathbb{Z}_3)$ by the simultaneous action of the three involutions $A \mapsto M_1 A$, $A \mapsto AM_1$, and $A \mapsto A^{-1}$. These involutions generate a group of order 8 isomorphic to a dihedral group. Hence,

$$Q\mathrm{Aut}(G)_{V_1,V_1}/(\Phi \mapsto \Phi^{-1}) \cong \mathrm{GL}_2(\mathbb{Z}_3)/D_4. \tag{1.7}$$

We have proved that families of regular product-quotient surfaces with quotient model $(C_1 \times C_2)/G$ where $G = S_3 \times \mathbb{Z}_3$, $\lambda_1 : C_1 \to \mathbb{P}^1$ and $\lambda_2 : C_2 \to \mathbb{P}^1$ are both ramified over three points are in bijection with $\mathrm{GL}_2(\mathbb{Z}_3)/D_4$, a set of cardinality 10. More precisely, these families are realized as follows: we consider two copies (C_1, ϕ), (C_2, ϕ) of the same curve (C, ϕ) defined in Example 1.10 which is described by the algebraic data V. This pair of curves define a product-quotient surface realizing a first family. All the other families are realized by product-quotient surfaces each defined by a pair (C_1, ϕ) and $(C_2, \phi \circ (\mathrm{Id}, A^{-1}))$, where A is a representative of a class of $\mathrm{GL}_2(\mathbb{Z}_3)/D_4$.

2. Finiteness of the classification problem

In this section, we follow step-by-step the same arguments of [4] and generalize the results of [4, Prop. 1.14] by removing the assumption $\chi = 1$ there.

As a byproduct, we describe an algorithm that produces for any fixed pair of positive integers K^2 and χ all regular product-quotient surfaces S of general type with self-intersection of the canonical class $K_S^2 = K^2$ and Euler characteristic $\chi(\mathcal{O}_S) = \chi$.

Let C_1 and C_2 be two Riemann surfaces of respective genera $g_1, g_2 \geq 2$ and let G be a finite group acting faithfully on both of them. We consider the diagonal action of G on the product $C_1 \times C_2$, which gives a product-quotient surface S, the minimal resolution of singularities of the quotient model $X := (C_1 \times C_2)/G$.

The singular points of the quotient model X are images of points in $C_1 \times C_2$ having non-trivial stabilizer by the diagonal action of G. Hence, X has only finitely many singular points which are cyclic quotient singularities.

A cyclic quotient singularity of type $\frac{1}{n}(1,a)$ is a singular point realized as the quotient of $\mathbb{C}^2$ by the action of the diagonal linear isomorphism of eigenvalues $\zeta_n = \exp \frac{2\pi i}{n}$ and ζ_n^a, with $\gcd(n,a) = 1$.

We can attach to X the so-called *basket* of singularities.

Definition 2.1 ([4, Def. 1.2]). A *representation of the basket of singularities of X* is a multiset

$$\mathcal{B}(X) := \left\{ \lambda \times \left(\frac{1}{n}(1,a) \right) : X \text{ has exactly } \lambda \text{ singularities of type } \frac{1}{n}(1,a) \right\}.$$

We use the word "representation" since X may have several representatives of its basket, essentially since a singularity of type $\frac{1}{n}(1,a)$ is isomorphic to a singularity of type $\frac{1}{n}(1,a')$, where either $a = a'$ or $aa' \equiv 1 \mod n$. This motivates the following definition.

Definition 2.2 ([4, Def. 1.4]). Consider the set of multisets of the form

$$\left\{\lambda \times \left(\frac{1}{n}(1,a)\right) : a, n, \lambda \in \mathbb{N}, a < n, \gcd(a,n) = 1\right\},$$

and define the equivalence relation given by "$\frac{1}{n}(1,a)$ *is equivalent to* $\frac{1}{n}(1,a')$" if $a = a'$ or $aa' \equiv 1 \mod n$. A basket of singularities is then an equivalence class.

In [4], the authors used the minimal resolution of a cyclic quotient singularity as *Hirzebruch–Jung string* to compute these correction terms to the self-intersection of the canonical class and the topological characteristic of the product-quotient surface S. We need to remember these correction terms.

Definition 2.3 ([4, Def. 1.5]). Let x be a singularity of type $\frac{1}{n}(1,a)$ with $\gcd(n,a) = 1$, and let $1 \le a' < n$ be the inverse of a modulo n, $a' = a^{-1}$. Write $\frac{n}{a}$ as a continued fraction

$$\frac{n}{a} = b_1 - \frac{1}{b_2 - \frac{1}{b_3 - \cdots}} = [b_1, \dots, b_l].$$

We define the following correction terms:

- $k_x := k(\frac{1}{n}(1,a)) = -2 + \frac{2+a+a'}{n} + \sum_{i=1}^{l}(b_i - 2) \ge 0$;
- $e_x := e(\frac{1}{n}(1,a)) = l + 1 - \frac{1}{n} \ge 0$;
- $B_x := 2e_x + k_x$.

Let $\mathcal{B}$ be the basket of singularities of X. We define

$$k(\mathcal{B}) := \sum_{x\in\mathcal{B}} k_x, \quad e(\mathcal{B}) := \sum_{x\in\mathcal{B}} e_x, \quad B(\mathcal{B}) := \sum_{x\in\mathcal{B}} B_x.$$

Theorem 2.4 ([4, Prop. 1.6 and Cor. 1.7]). *Let $\rho\colon S \to X$ be the minimal resolution of the singularities of $X = (C_1 \times C_2)/G$. Then, the self-intersection of the canonical class of S and its topological Euler characteristic are equal to*

$$K_S^2 = \frac{8(g_1-1)(g_2-1)}{|G|} - k(\mathcal{B}), \quad \textit{and} \quad e(S) = \frac{4(g_1-1)(g_2-1)}{|G|} + e(\mathcal{B}).$$

Furthermore, it holds that

$$K_S^2 = 8\chi(\mathcal{O}_S) - \frac{1}{3}B(\mathcal{B}).$$

From now on we shall restrict to product-quotient surfaces S of general type which are regular, namely, $C_i/G \cong \mathbb{P}^1$.

As explained in Section 1.2, we shall describe S in a pure algebraic way by using a pair of spherical systems of generators

$$[g_1, \dots, g_r] \quad \text{and} \quad [h_1, \dots, h_s]$$

of the pair of G-coverings C_1 and C_2 of $\mathbb{P}^1$.

Remark 2.5. In [4, Sec. 1.2] is shown how to determine the number of cyclic quotient singularities (and their types) of the quotient model $X = (C_1 \times C_2)/G$ from a pair of spherical systems of generators.

In this way, we read the basket of singularities of S from the pair $[g_1, \dots, g_r]$ and $[h_1, \dots, h_s]$, and then determine the invariants K_S^2 and $\chi(\mathcal{O}_S)$ by using Theorem 2.4.

We state the preliminaries to extend [4, Prop. 1.14] to any positive integer χ.

Definition 2.6. Fix an r-tuple of natural numbers $t := [m_1, \dots, m_r]$, and a basket of singularities $\mathcal{B}$. Then, we associate with these the following numbers:

$$\Theta(t) := -2 + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right); \quad \alpha(t, \mathcal{B}, \chi) := \frac{12\chi + k(\mathcal{B}) - e(\mathcal{B})}{6\Theta(t)}.$$

We recall the following definition.

Definition 2.7. The minimal positive integer I_x such that $I_x K_X$ is Cartier in x is called the *index* of the singularity x.

The index of X is the minimal positive integer I such that IK_X is Cartier. In particular, $I = \operatorname{lcm}_{x \in \operatorname{Sing} X} I_x$.

As remarked in [4], the index of a cyclic quotient singularity $\frac{1}{n}(1, a)$ is

$$I_x = \frac{n}{\gcd(n, a+1)}.$$

By [4, Lem. 1.10], fixing a pair of positive integers (K^2, χ), there are only finitely many baskets of singularities $\mathcal{B}$ for which there exists a product-quotient surface S with invariants $K_S^2 = K^2$, $\chi(\mathcal{O}_S) = \chi$, and having a quotient model with a representation of the basket of singularities equal to $\mathcal{B}$.

We need to extend [4, Prop. 1.14] to any positive integer χ to bound, for fixed K^2, χ, and $\mathcal{B}$, the possibilities for

- $|G|$,
- $t_1 := [m_1, \dots, m_r]$,
- $t_2 := [n_1, \dots, n_s]$,

of a product-quotient surface S with $K_S^2 = K^2$, $\chi(S) = \chi$, and basket of singularities of the quotient model $X = (C_1 \times C_2)/G$ equal to $\mathcal{B}$ such that the pair of spherical systems of generators of C_1 and C_2 have, respectively, signature t_1 and t_2.

PROPOSITION 2.8. *Fix a pair $(K^2, \chi) \in \mathbb{Z} \times \mathbb{Z}$, and fix a possible basket of singularities $\mathcal{B}$ for (K^2, χ). Let S be a product-quotient surface of general type such that*

(i) *$K_S^2 = K^2$;*

(ii) *$\chi(S) = \chi$;*

(iii) *the basket of singularities of the quotient model $X = (C_1 \times C_2)/G$ equals $\mathcal{B}$.*

Then,

(a) *$g(C_1) = \alpha(t_2, \mathcal{B}, \chi) + 1$, $g(C_2) = \alpha(t_1, \mathcal{B}, \chi) + 1$;*

(b) *$|G| = \frac{8\alpha(t_1,\mathcal{B},\chi)\alpha(t_2,\mathcal{B},\chi)}{K^2+k(\mathcal{B})}$;*

(c) *$r, s \leq \frac{K^2+k(\mathcal{B})}{2} + 4$;*

(d) *m_i divides $2\alpha(t_1, \mathcal{B}, \chi)I$, and n_j divides $2\alpha(t_2, \mathcal{B}, \chi)I$;*

(e) *there are at most $|\mathcal{B}|/2$ indices i such that m_i does not divide $\alpha(t_1, \mathcal{B}, \chi)$, and similarly for the n_j;*

(f) *$m_i \leq \frac{1+I\frac{K^2+k(\mathcal{B})}{2}}{f(t_1)}$, $n_i \leq \frac{1+I\frac{K^2+k(\mathcal{B})}{2}}{f(t_2)}$, where I is the index of X, and $f(t_1) := \max(\frac{1}{6}, \frac{r-3}{2})$, $f(t_2) := \max(\frac{1}{6}, \frac{s-3}{2})$;*

(g) *except for at most $|\mathcal{B}|/2$ indices i, the sharper inequality $m_i \leq \frac{1+\frac{K^2+k(\mathcal{B})}{4}}{f(t_1)}$ holds, and similarly for the n_j.*

REMARK 2.9 ([4, Rem. 1.15]). Note that (b) shows t_1 and t_2 determine the order of G. (c) and (f) imply there are only finitely many possibilities for the signatures t_1, t_2. Instead, (d), (e), and (g) are strictly necessary to obtain an efficient algorithm.

PROOF. The proof is analogous to the one in [4, Prop. 1.14]. ■

2.1. *Description of the classification algorithm*

Fixing a pair $(K^2, \chi) \in \mathbb{N} \times \mathbb{N}$, the next goal is to write a MAGMA script to find all minimal regular surfaces S of general type with $K_S^2 = K^2$, and $\chi(S) = \chi$, which are product-quotient surfaces. A commented version of the MAGMA code is available here.

We describe here the strategy, and explain how the most important scripts work. Most of the scripts are the modification of those in [4]. Since those scripts were written under the assumption $\chi = 1$, we generalize all of them to allow any value of χ. In the Introduction of the present paper, we indicate the other main improvements we did.

We fix a couple (K^2, χ). Note that by the minimality of S, and by Theorem 2.4, then $K^2 \in \{1, \dots, 8\chi\}$, and the case $K^2 = 8\chi$ corresponds to surfaces isogenous to a product.

STEP 1. The script *Baskets* lists all the *possible baskets of singularities* $\mathcal{B}$ for (K^2, χ). Indeed, there are only finitely many of them by [4, Lem. 1.10]. The input is $B(\mathcal{B}) = 3(8\chi - K^2)$, so to get for instance all baskets for $(K^2, \chi) = (28, 4)$, we need *Baskets*(12).

STEP 2. From Proposition 2.8, once we know the basket of singularities of $X = (C_1 \times C_2)/G$, then there are finitely many possible signatures of a pair of spherical systems of generators of C_1 and C_2. *ListOfTypes* computes them using the inequalities in Proposition 2.8. Here, the input is K^2, and χ, so *ListOfTypes* first computes *Baskets*$(3(8\chi - K^2))$, and then computes for each basket all numerically compatible signatures. The output is a list of pairs, the first element of each pair being a basket, and the second element being the list of all signatures compatible with that basket.

STEP 3. Every surface produces two signatures, one for each curve C_i, both compatible with the basket of singularities of X; if we know the signatures and the basket, then Proposition 2.8 (b) tells us the order of G. *ListGroups*, whose input is K^2, and χ, first computes *ListOfTypes*(K^2, χ). Then, for each pair of signatures in the output, it determines the order of the group. Next, it searches among the groups of a given order whose groups admit appropriate spherical systems of generators corresponding to both signatures. Here, we use the database in [15] if we are in one of the cases classified there; otherwise, we use the function *FindGenerators* developed in the work [15].

For each affirmative answer, it stores the triple (basket, pair of signatures, group) in a list, which is the main output.

The script has some shortcuts:

- Let t_1 and t_2 be the pair of signatures and let $\mathbb{T}(t_1)$ and $\mathbb{T}(t_2)$ be their respective polygonal groups (see Remark 1.5). Then, the order of the abelianization G^{ab} of G has to divide the order of the abelianization of $\mathbb{T}(t_1)$ and $\mathbb{T}(t_2)$:

$$|G^{ab}| \text{ divides } \left|\mathbb{T}(t_1)^{ab}\right|, \left|\mathbb{T}(t_2)^{ab}\right|. \tag{2.1}$$

  Indeed, the appropriate orbifold (surjective) homomorphisms $\mathbb{T}(t_1) \to G$ and $\mathbb{T}(t_2) \to G$ induce surjective homomorphisms

$$\mathbb{T}(t_1)^{ab} \to G^{ab}, \quad \mathbb{T}(t_2)^{ab} \to G^{ab}.$$

  Hence, *ListGroups* checks first if G satisfies (2.1): if not, this case does not occur.
- If the pair of signatures t_1 and t_2 returns polygonal groups $\mathbb{T}(t_1)$ and $\mathbb{T}(t_2)$ such that the orders of their abelianization are coprime numbers, then G is forced to be a perfect group. This follows directly from the condition (2.1).

MAGMA knows all perfect groups of order ≤ 50000, and then *ListGroups* checks first if there are perfect groups of the right order: if not, this case cannot occur.

- If
  - either the expected order of the group is 1024 or bigger than 2000, which are not in the MAGMA database of finite groups,
  - or the order is a number as, e.g., 1728, where there are too many isomorphism classes of groups,

  then *ListGroups* just stores these cases in a list, secondary output of the script. These "exceptional" cases have to be considered separately.

STEP 4. *ExistingSurfaces* runs on the output of *ListGroups*(K^2, χ) and throws away all triples giving rise only to surfaces whose singularities do not correspond to the basket.

STEP 5. Each triple (basket, pair of signatures, group) belonging to the output *ExistingSurfaces*(K^2, χ) gives many different pairs of compatible spherical systems of generators. On them there is the action of $\mathrm{Aut}(G) \times \mathcal{B}_r \times \mathcal{B}_s$ described in Section 1.2. Therefore, *FindSurfaces* uses Theorem 1.20 and Corollary 1.22 to pick up only one pair of spherical systems of generators for each family of product-quotient surfaces compatible with the triple (basket, pair of signatures, group). Thus, the output is a list of (basket, sph1, sph2, group), where sph1 and sph2 are spherical systems of group compatible with the two signatures and the basket.

3. CLASSIFICATION OF REGULAR PRODUCT-QUOTIENT SURFACES WITH $23 \leq K^2 \leq 32$ AND $\chi = 4$

In this section, we prove the main Theorem 0.3 presented in the introduction.

We have run the function *FindSurfaces* described in Section 2.1 on each triple of the output of *ExistingSurfaces*(K^2, χ), where $K^2 \in \{23, \dots, 32\}$ and $\chi = 4$. This has given all the families in Tables 9 to 21 of the appendix with the only exception of families no. 267 and 544, which are the only cases that occurred on those skipped by *ListGroups* and stored in its secondary output.

Thus, to prove the main Theorem 0.3, it remains to show that

(1) among all the exceptional cases skipped by *ListGroups*, only two cases occur, which are no. 267 and 544;

(2) all the obtained families of Tables 9 to 21 are of general type and those on Tables 9 to 20 are also minimal.

This will be the content of Sections 3.1 and 3.2.

3.1. *The exceptional cases*

For each $K^2 \in \{23, \dots, 32\}$, the list of cases skipped by *ListGroups*$(K^2, 4)$ and stored in its secondary output can be found here.

We present the main theorem of this subsection.

THEOREM 3.1. *There are exactly two groups G admitting an appropriate pair of spherical systems of generators compatible with one of the triples of the secondary output of ListGroups$(K^2, 4)$, for $K^2 \in \{23, \dots, 32\}$:*

No.	K_S^2	$\mathrm{Sing}(X)$	t_1	t_2	G	N
267	26	$1/4, 1/2^2, 3/4$	$3^2, 4$	$3^2, 4$	$G(1944, 3875)$	2
544	24	$1/6, 1/2^2, 5/6$	$2, 4, 6$	$2, 6, 8$	$G(768, 1086051)$	2

A proof of this theorem can be found in the *HowToRemoveTocheck.txt* files on the webpage linked above, with one file for each $K^2 \in \{23, \dots, 32\}$. More precisely, these files provide a step-by-step explanation of how to exclude the cases omitted by *ListGroups* until only the two cases mentioned above are found to actually occur.

However, to illustrate the main strategy we have employed to exclude these cases, here we only discuss those with $K^2 = 32$, which already consist of a significant list of 152 cases. Therefore, we need to prove the following theorem.

THEOREM 3.2. *No one of the cases skipped by ListGroups*$(32, 4)$ *gives a product-quotient surface S with $K_S^2 = 32$ and $\chi(\mathcal{O}_S) = 4$.*

PROOF. It follows from Propositions 3.5, 3.8, 3.10, and Remark 3.13 below. ■

The rest of this section is devoted to giving a proof of the series of propositions used to prove Theorem 3.2.

We use two MAGMA functions to prove these propositions and more in general Theorem 3.1:

- *HowToExclude* takes in input a list of triples as those of the second output of *ListGroups* which have an order of the group different from 1024 and less than or equal to 2000. For each triple (basket, (t_1, t_2), *ord*) of the list, it returns those groups with order *ord* admitting a pair of spherical systems of generators of signatures t_1 and t_2. This function uses as *ListGroups* the database and function *FindGenerators* in [15].
- The function *HowToExcludePG* works similarly such as *HowToExclude*. Hence, it takes in input a list of triples (basket, (t_1, t_2), *ord*), where *ord* is ≤ 50000, and returns those groups with order *ord* that are perfect and admit a pair of spherical systems of generators of signatures t_1 and t_2.

REMARK 3.3. To exclude the cases skipped by *ListGroups*(32, 4), and more generally the cases mentioned in Theorem 3.1, the main strategy is to assume their existence by contradiction and construct from them a new product-quotient surface by a smaller normal subgroup $H \trianglelefteq G$ with a new pair of signatures derived from the previous data. This process is repeated until the order of the group H becomes sufficiently small to apply the code *HowToExclude* on H and the new pair of signatures. If the code excludes this case, then the initial case must also be excluded. Otherwise, we run *ExistingSurfaces* to verify that the basket of singularities of the new product-quotient surface is compatible with the basket of singularities of the initial case.

Note that the basket of singularities for the new surfaces constructed at each intermediate step is always empty when the initial case has an empty basket, as in the case with $K^2 = 32$. For this reason, we will avoid repeating the basket of singularities for intermediate steps in such cases, as it remains empty throughout.

On the other hand, for the remaining cases with $K^2 \in \{23, \ldots, 30\}$, as discussed in the .txt files on the webpage, we point out that the intermediate steps may involve a non-empty basket of singularities. In these situations, we must run *ExistingSurfaces* for the group H and the new pair of signatures, considering all possible intermediate baskets compatible with the initial one.

Notation. Given positive integers a and p, the expression a^p represents the sequence consisting of the same element a repeated p times. For example, the sequence $[3^2, 4^3]$ corresponds to $[3, 3, 4, 4, 4]$.

PROPOSITION 3.4. *Let G be a finite group that admits a spherical system of generators of signature $[a_1, a_2, a_3, b_1, \ldots, b_k]$. Let us suppose G have a normal subgroup H of index a prime number $p \geq 2$ and that p does not divide $b_1, \ldots, b_k$. Then,*

- *if p does not divide only one among a_1, a_2, a_3, e.g. $p \nmid a_3$, then H admits a spherical system of generators of signature $[a_1/p, a_2/p, a_3^p, b_1^p, \ldots, b_k^p]$;*
- *if p divides each one of a_1, a_2, a_3, then H admits either a spherical system of generators having one of the following signatures:*

  (1) $[a_1/p, a_2/p, a_3^p, b_1^p, \ldots, b_k^p]$;

  (2) $[a_1/p, a_2^p, a_3/p, b_1^p, \ldots, b_k^p]$;

  (3) $[a_1^p, a_2/p, a_3/p, b_1^p, \ldots, b_k^p]$;

  or if $p \neq 2$, then there exists H-covering of a curve of genus $\frac{p-1}{2}$ whose branch locus has ramification indices $a_1/p, a_2/p, a_3/p, b_1^p, \ldots, b_k^p$.

PROOF. By assumption, G has a spherical system of generators $[g_1, g_2, g_3, h_1 \ldots, h_k]$ which defines a G-covering $C \to \mathbb{P}^1$ whose branch locus $v_1, v_2, v_3, q_1, \ldots, q_k \in \mathbb{P}^1$

has ramification indices $a_1, a_2, a_3, b_1, \dots, b_k$, respectively. Furthermore, the existence of a normal subgroup H of index p gives the following triangular commutative diagram:

$$\begin{tikzcd} C \arrow[d, "/H"'] \arrow[dr, "/G"] & \\ C/H \arrow[r, "/\mathbb{Z}_p"'] & \mathbb{P}^1. \end{tikzcd}$$

Note that $h_i \in H$ since $h_i H$ has order in $G/H \cong \mathbb{Z}_p$ that divides both p and the order b_i of h_i. Hence, $q_1, \dots, q_k$ are not in the branch locus of $C/H \to \mathbb{P}^1$, which has then to branch over at most $r \leq 3$ points with ramification indices p.

By Hurwitz formula (1.2), we get

$$2g(C/H) - 2 = p\left(-2 + r\frac{p-1}{p}\right) \implies g(C/H) = \frac{p-1}{2}(r-2). \tag{3.1}$$

Hence, r is forced to be equal to either 2 or 3. If $r = 2$, then $C/H \cong \mathbb{P}^1$, and we can assume without lost of generality that v_3 is not in the branch locus, so in other words, $g_3 \in H$.

We want to determine the signature of a spherical system of generators that defines $C \to C/H \cong \mathbb{P}^1$. Each point of the fibre of q_i via $C/H \to \mathbb{P}^1$ is contained in the branch locus of $C \to C/H$ and has ramification index b_i since $h_i \in H$. Note that the cardinality of the fibre is exactly p for these points q_i. The same holds for v_3 since g_3 belongs to H.

Instead, the fibre of v_i on C/H consists of only one point, $i = 1, 2$. The ramification index of this point for $C \to C/H$ equals the order of $\langle g_i \rangle \cap H$, which is a_i/p. We therefore obtain the signature $[a_1/p, a_2/p, a_3^p, b_1^p, \dots, b_k^p]$.

The case $r = 3$ can be discussed by using the same argument. ∎

Proposition 3.5. *There are exactly five groups G of order different from* 1024 *and less than or equal to 2000 admitting an appropriate pair of spherical systems of generators compatible with one of the triples of the secondary output of ListGroups*$(32, 4)$*:*

t_1	t_2	G
$2, 4, 6$	$2^3, 4$	$G(768, 1086051)$
$2, 4, 6$	$2^3, 4$	$G(768, 1086052)$
$2, 4, 6$	$2, 4, 20$	$G(960, 5719)$
$2, 4, 6$	$2, 4, 12$	$G(1152, 157849)$
$2, 4, 5$	$2, 4, 12$	$G(1920, 240996)$

However, no one of these cases gives product-quotient surfaces isogenous to a product.

PROOF. We collect in a list those triples of the secondary output of *ListGroups*$(32, 4)$ having an order of the group different from 1024 and less than or equal to 2000. Then, we run *HowToExclude* on this list and we obtain the above table.

However, we use *ExistingSurfaces* for each of the rows of the table to check that no one gives a product-quotient surface isogenous to a product. ■

As a consequence of the previous statement, it remains to discuss only 65 of 152 cases skipped by *ListGroups*, which are those of Tables 2 and 3 below.

REMARK 3.6. From (2.1), we get that groups G having group order and a pair of spherical system of generators compatible with one of the rows of Tables 2 and 3 satisfy the following:

(1) from no. 1 to no. 18 are perfect groups;
(2) from no. 19 to no. 54 are either perfect groups or $G^{ab} \cong \mathbb{Z}_2$;
(3) from no. 55 to no. 62 are either perfect groups or $G^{ab} \cong \mathbb{Z}_3$;
(4) either no. 63 is a perfect group or G^{ab} is isomorphic to $\mathbb{Z}_2$ or to $\mathbb{Z}_2 \times \mathbb{Z}_2$;
(5) either no. 64 is perfect or G^{ab} is isomorphic to $\mathbb{Z}_2$ or to $\mathbb{Z}_3$ or to $\mathbb{Z}_6$;
(6) either no. 65 is perfect or G^{ab} is isomorphic to one among $\mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_4, \mathbb{Z}_4 \times \mathbb{Z}_2$.

LEMMA 3.7. *There are no perfect groups G having group order and a pair of spherical systems of generators of signatures compatible with one of the rows of Tables 2 and 3.*

PROOF. We use *HowToExcludePG* on the list of triples of Tables 2 and 3 to check that there are no perfect groups having compatible algebraic data. ■

PROPOSITION 3.8. *There are no groups G having group order and a pair of spherical systems of generators of signatures compatible with one of the rows of Tables 2 and 3 from no. 1 to no. 18.*

PROOF. This follows directly from Remark 3.6 and Lemma 3.7. ■

We consider now rows from no. 19 to no. 62 of Tables 2 and 3.

REMARK 3.9. We need the following classical remarks of group theory:

(1) Let G be a finite group having a normal subgroup H of index a prime number $p \geq 2$. If there is an element $g \in G$, $g \notin H$, of order p, then
$$0 \to H \to G \to \mathbb{Z}_p \to 0$$
is a split exact sequence via the homomorphism section sending $\bar{1} \in \mathbb{Z}_p$ to g. In other words, $G = H \rtimes_\phi \mathbb{Z}_p$, where ϕ is an automorphism of H of order p.

(2) Let $\pi : G \to Z$ be a surjective group homomorphism. If Z admits a normal subgroup T of index $k \in \mathbb{N}$, then $H := \pi^{-1}(T)$ is a normal subgroup of G of index k. More precisely, $G/H \cong Z/T$.

No.	t_1	t_2	$\lvert G\rvert$
1	$2,3,8$	$2,5^2$	3840
2	$2,3,7$	$4,4,4$	2688
3	$2,3,7$	$2,3,18$	6048
4	$2,3,7$	$2,4,8$	5376
5	$2,3,7$	$3,3,5$	5040
6	$2,3,7$	$2,5,6$	5040
7	$2,3,7$	$2,8,8$	2688
8	$2,3,7$	$3,3,15$	2520
9	$2,3,7$	$2,3,7$	28224
10	$2,3,7$	$2,5,30$	2520
11	$2,3,7$	$2,3,10$	10080
12	$2,3,7$	$2,2,2,4$	2688
13	$2,3,7$	$2,6,15$	2520
14	$2,3,7$	$3,5,5$	2520
15	$2,3,7$	$2,3,30$	5040
16	$2,4,5$	$3,3,4$	3840
17	$2,3,9$	$2,4,5$	5760
18	$2,3,9$	$2,5,6$	2160
19	$2,3,12$	$2,4,6$	2304
20	$2,3,10$	$2,4,6$	2880
21	$2,3,8$	$2,4,12$	2304
22	$2,3,8$	$2,5,6$	2880
23	$2,3,22$	$2,4,5$	2640
24	$2,3,12$	$2,4,5$	3840
25	$2,3,14$	$2,4,6$	2016
26	$2,3,8$	$2,4,6$	4608
27	$2,3,18$	$2,4,5$	2880
28	$2,3,10$	$2,4,5$	4800
29	$2,3,54$	$2,4,5$	2160
30	$2,4,5$	$2,4,6$	3840
31	$2,3,30$	$2,4,5$	2400
32	$2,4,5$	$2,4,8$	2560
33	$2,3,8$	$2,4,5$	7680

TABLE 2.

No.	t_1	t_2	$\lvert G\rvert$
34	$2,3,14$	$2,4,5$	3360
35	$2,3,8$	$2,4,8$	3072
36	$2,3,8$	$2,6,7$	2016
37	$2,3,10$	$2,3,10$	3600
38	$2,3,8$	$2,3,18$	3456
39	$2,3,8$	$2,3,54$	2592
40	$2,4,5$	$2,5,6$	2400
41	$2,3,8$	$2,3,22$	3168
42	$2,3,12$	$2,3,14$	2016
43	$2,3,8$	$2,3,30$	2880
44	$2,3,8$	$2,2,2,3$	2304
45	$2,3,8$	$2,6,6$	2304
46	$2,3,8$	$3,4,4$	2304
47	$2,3,10$	$2,3,18$	2160
48	$2,3,10$	$2,3,14$	2520
49	$2,3,10$	$2,3,12$	2880
50	$2,3,8$	$2,3,14$	4032
51	$2,3,8$	$2,3,8$	9216
52	$2,4,5$	$2,4,5$	6400
53	$2,3,8$	$2,3,12$	4608
54	$2,3,8$	$2,3,10$	5760
55	$2,3,9$	$3,3,5$	2160
56	$2,3,9$	$2,3,12$	3456
57	$2,3,12$	$3,3,4$	2304
58	$2,3,9$	$2,3,18$	2592
59	$2,3,9$	$2,3,30$	2160
60	$2,3,9$	$2,3,9$	5184
61	$3,3,4$	$3,3,4$	2304
62	$2,3,9$	$3,3,4$	3456
63	$2,4,6$	$2,4,6$	2304
64	$2,3,12$	$2,3,12$	2304
65	$2,4,8$	$2,4,8$	1024

TABLE 3.

No.	t_1	t_2	$\|G'\|$
19(a)	3, 3, 6	2, 2, 2, 3	1152
(b)	3, 3, 6	3, 4, 4	1152
(c)	3, 3, 6	2, 6, 6	1152
20(a)	3, 3, 5	2, 2, 2, 3	1440
(b)	3, 3, 5	3, 4, 4	1440
(c)	3, 3, 5	2, 6, 6	1440
21(a)	3, 3, 4	2, 2, 2, 6	1152
(b)	3, 3, 4	4, 4, 6	1152
(c)	3, 3, 4	2, 12, 12	1152
22	3, 3, 4	3, 5, 5	1440
23	3, 3, 11	2, 5, 5	1320
24	3, 3, 6	2, 5, 5	1920
25(a)	3, 3, 7	2, 2, 2, 3	1008
(b)	3, 3, 7	3, 4, 4	1008
(c)	3, 3, 7	2, 6, 6	1008
26(a)	3, 3, 4	2, 2, 2, 3	2304
(b)	3, 3, 4	3, 4, 4	2304
(c)	3, 3, 4	2, 6, 6	2304
27	3, 3, 9	2, 5, 5	1440
28	3, 3, 5	2, 5, 5	2400
29	3, 3, 27	2, 5, 5	1080
30(a)	2, 5, 5	2, 2, 2, 3	1920
(b)	2, 5, 5	3, 4, 4	1920
(c)	2, 5, 5	2, 6, 6	1920
31	3, 3, 15	2, 5, 5	1200
32(a)	2, 5, 5	2, 2, 2, 4	1280
(b)	2, 5, 5	4, 4, 4	1280
(c)	2, 5, 5	2, 8, 8	1280
33	3, 3, 4	2, 5, 5	3840
34	3, 3, 7	2, 5, 5	1680

TABLE 4.

No.	t_1	t_2	$\|G'\|$
35(a)	3, 3, 4	2, 2, 2, 4	1536
(b)	3, 3, 4	4, 4, 4	1536
(c)	3, 3, 4	2, 8, 8	1536
36	3, 3, 4	3, 7, 7	1008
37	3, 3, 5	3, 3, 5	1800
38	3, 3, 4	3, 3, 9	1728
39	3, 3, 4	3, 3, 27	1296
40	2, 5, 5	3, 5, 5	1200
41	3, 3, 4	3, 3, 11	1584
42	3, 3, 6	3, 3, 7	1008
43	3, 3, 4	3, 3, 15	1440
44	3, 3, 4	2, 2, 3, 3	1152
45(a)	3, 3, 4	2, 2, 3, 3	1152
(b)	3, 3, 4	3, 6, 6	1152
46	3, 3, 4	2, 2, 3, 3	1152
47	3, 3, 5	3, 3, 9	1080
48	3, 3, 5	3, 3, 7	1260
49	3, 3, 5	3, 3, 6	1440
50	3, 3, 4	3, 3, 7	2016
51	3, 3, 4	3, 3, 4	4608
52	2, 5, 5	2, 5, 5	3200
53	3, 3, 4	3, 3, 6	2304
54	3, 3, 4	3, 3, 5	2880
55	2, 2, 2, 3	5, 5, 5	720
56	2, 2, 2, 3	2, 2, 2, 4	1152
57	2, 2, 2, 4	4, 4, 4	768
58	2, 2, 2, 3	2, 2, 2, 6	864
59	2, 2, 2, 3	2, 2, 2, 10	720
60	2, 2, 2, 3	2, 2, 2, 3	1728
61	4, 4, 4	4, 4, 4	768
62	2, 2, 2, 3	4, 4, 4	1152

TABLE 5.

PROPOSITION 3.10. *There are no groups G having group order and a pair of spherical systems of generators defining a product-quotient surface isogenous to a product and compatible with one of the triples from no. 19 to no. 62 of Tables 2 and 3.*

PROOF. From Remark 3.6 and Lemma 3.7, groups G from no. 19 to no. 62 of Tables 2 and 3 have a commutator subgroup $G' := [G, G]$ of index equal to either 2 or 3. Hence, we can apply Proposition 3.4 to $H = G'$ and say that G' has group order and a pair of spherical systems of generators compatible with one of the triples of Tables 4 and 5.

REMARK 3.11. Running *HowToExcludePG* on the list of Tables 4 and 5, we see that there are no compatible perfect groups G'.

From (2.1), we see that triples of Tables 4 and 5 from no. 19 to no. 36 (with the exception of no. 19(c), 20(c), 21(c), 25(c), 26(c)) together with no. 55 have G' forced to be a perfect group, which is a contradiction with Remark 3.11.

We run *HowToExclude* on no. 19(c), 20(c), 21(c), 25(c) to prove that there are no groups compatible with those algebraic data.

Instead, we exclude 26(c) using the following.

REMARK 3.12 ([8, Lem. 4.11]). There are no groups of order 768 having a spherical system of generators of signature $[4, 4, 4]$.

Indeed, we would get $G'' = [G', G']$ of 26(c) is a group of order 768 and from Proposition 3.4 it should admit a spherical system of generators of signature $[4, 4, 4]$.

We have excluded all cases from no. 19 to no. 36 together with no. 55 of Tables 2 and 3.

We conclude by discussing no. 52 only, as all remaining cases in Table 3 can be discarded using analogous arguments.

We recall Remark 3.11 and so we apply Proposition 3.4 to the commutator $G'' \lhd G'$, which has then order 640 and admits a pair of spherical systems of generators both with signature 2^5.

We run *HowToExclude* and then *ExistingSurfaces* to see that there are only four groups $G''(640, n)$ having a pair of spherical systems of generators with signature 2^5 defining a product-quotient surface isogenous to product, where $n = 7665, 8697, 12278, 15814$.

However, G'' has index 5 in G', which admits a spherical system of generators $[g_1, g_2, g_3]$ of signature $[2, 5, 5]$. Then, $g_2 \notin G''$ and it has order 5. This means from Remark 3.9 (1) that

$$0 \to G'' \to G' \to \mathbb{Z}_5 \to 0$$

is a splitting exact sequence, so $G' = G'' \rtimes_\phi \mathbb{Z}_5$ through an automorphism ϕ of G'' of order 5. We easily check that each of the obtained groups $G''(640, n)$ admits exactly four automorphisms of order 5. However, for each of these automorphisms ϕ the semidirect product $G''(640, n) \rtimes_\phi \mathbb{Z}_5$ has abelianization $\mathbb{Z}_2^4 \times \mathbb{Z}_5$, so no one of these groups can be G' of no. 52 in Table 5, which has abelianization $\mathbb{Z}_5$.

This then excludes groups G of no. 52 of Table 3. ■

Finally, we are left to consider rows no. 63, 64, 65 of Table 3.

REMARK 3.13. Using similar arguments as in the proof of Proposition 3.10, groups G of the rows no. 63, 64, 65 do not yield surfaces isogenous to a product, and so these cases can be discarded.

3.2. *Rational* (-1)*-curves on product-quotient surfaces*

In this short subsection, we investigate which surfaces among those obtained in Theorem 0.3 do not contain (-1)-curves, namely smooth rational curves with self-intersection -1. First of all, we observe the following.

Remark 3.14. All surfaces S obtained in Theorem 0.3 are surfaces of general type. Indeed, from Enriques–Kodaira classification of complex algebraic surfaces, if $q(S)$ is zero, then either S is rational, or S is of general type, or $K_S^2 \leq 0$. Therefore, since surfaces of Theorem 0.3 have $K_S^2 \geq 23$, $q(S) = 0$, and $p_g(S) = 3 \neq 0$, then they are of general type.

Proposition 3.15 ([5, Lem. 6.9]). *Let S be a product-quotient surface of general type of quotient model X. Assume that the exceptional locus of the minimal resolution of singularities $\rho\colon S \to X$ consists of*

(i) *curves of self-intersection (-3) and (-2), or*

(ii) *at most two smooth rational curves of self-intersection (-3) or (-4), and (-2)-curves.*

Then, S is minimal, so it does not contain (-1)-curves.

Corollary 3.16. *Let S be a product-quotient surface belonging to one of the families of Tables 9 to 20 of Theorem 0.3. Then, S is a minimal surface.*

Proof. For each case of Tables 9 to 20 (with the exception of no. 186 to 196), the exceptional curves arising from the basket of singularities of the quotient model X are either of type (i) or (ii) of Proposition 3.15, so that S is minimal.

Regarding the remaining cases no. 186 to 196, their basket of singularities is always equal to $\{1/5, 4/5\}$, so the minimality follows directly by [4, Prop. 4.7 (3)]. ■

4. The degree of the canonical map of product-quotient surfaces

In this section, we investigate the degree of the canonical map of product-quotient surfaces, with a particular focus on those having geometric genus three.

We briefly explain the strategy and the content of each subsection but first we give the following.

Remark 4.1. The degree of the canonical map of product-quotient surfaces is bounded from above by 32. Indeed, product-quotient surfaces satisfy the inequality $K_S^2 \leq 8\chi(\mathcal{O}_S)$, see Theorem 2.4, and so replacing Bogomolov–Miyaoka–Yau inequality with $K_S^2 \leq 8\chi(\mathcal{O}_S)$ in the proof of [9, Prop. 4.1], we get

$$\deg(\Phi_S) \leq \frac{8\chi(\mathcal{O}_S)}{\chi(\mathcal{O}_S) - 3} \leq 32.$$

Let us consider a product-quotient surface S given by a pair of curves C_1 and C_2 and a finite group G acting (faithfully) on both of them.

The diagonal action of G on the product $C_1 \times C_2$ induces a representation of G on the spaces of 2-forms of $C_1 \times C_2$. Let us denote by $|K_{C_1\times C_2}|^G$ the linear subsystem of the canonical linear system of $C_1 \times C_2$ given by the subspace $H^{2,0}(C_1 \times C_2)^G$ of the G-invariant 2-forms.

In Section 4.4, we show the relationship between the degree of the canonical map of S and the (schematic) base locus of the moving part of $|K_{C_1\times C_2}|^G$. Indeed, it holds that

$$\deg(\Phi_S) = \frac{1}{|G| \cdot \deg(\Sigma)} \cdot \widehat{M}^2, \tag{4.1}$$

where Σ is the image of the canonical map of S, and $\widehat{M}$ is the base-point free linear system obtained blowing-up the base locus of the moving part of $|K_{C_1\times C_2}|^G$.

Note that whenever $p_g(S) = 3$, then the image of the canonical map is $\mathbb{P}^2$, a surface of degree 1, and so the knowledge of the base locus of $|K_{C_1\times C_2}|^G$ is enough to compute $\deg(\Phi_S)$ by using Formula (4.1).

The strategy to investigate the base locus of $|K_{C_1\times C_2}|^G$ is the following. The action of G induces a representation on the space of 1-forms $H^{1,0}(C_i)$ via pullback, called in the literature *canonical representation*. By the standard representation theory, the space of 1-forms splits as a direct sum of isotypic components $H^{1,0}(C_i)^\chi$, $\chi \in \mathrm{Irr}(G)$ irreducible character of G. The irreducible characters χ occurring in the character χ_{can} of the canonical representation are explicitly computable by the Chevalley–Weil formula, see [19, Thm. 2.8].

As a consequence of this, the space of invariant 2-forms $H^{2,0}(C_1 \times C_2)^G$ splits as a direct sum of invariant subspaces

$$\left(H^{1,0}(C_1)^\chi \otimes H^{1,0}(C_2)^{\bar\chi}\right)^G, \quad \chi \in \mathrm{Irr}(G).$$

Therefore, the base locus of $|K_{C_1\times C_2}|^G$ is simply the intersection of the base loci of such invariant subspaces and then a computation of them solves the problem.

Let us consider the natural inclusion

$$\left(H^{1,0}(C_1)^\chi \otimes H^{1,0}(C_2)^{\bar\chi}\right)^G \subseteq H^{1,0}(C_1)^\chi \otimes H^{1,0}(C_2)^{\bar\chi}. \tag{4.2}$$

Theorem 4.20 determines the base locus of the linear subsystem of the bigger subspace $H^{1,0}(C_1)^\chi \otimes H^{1,0}(C_2)^{\bar\chi}$, which is discovered to be pure in codimension 1 and union of fibres (with multiplicities) for the natural projections $C_1 \times C_2 \to C_i$, $i = 1, 2$.

The formula to compute explicitly these fibres and their multiplicities is given through Theorem 4.10 which provides the base locus of the subsystem of the canonical system of a Riemann surface C given by an isotypic component $H^{1,0}(C)^\chi$ of the action of a finite group G on C.

Note that whenever χ is of degree one, then (4.2) is an equality. This motivates the following.

Property (#) A product-quotient surface S satisfies Property (#) if

$$\dim\left(H^{1,0}(C_1)^\chi \otimes H^{1,0}(C_2)^{\overline{\chi}}\right)^G \neq 0 \implies \deg(\chi) = 1$$

for each $\chi \in \mathrm{Irr}(G)$.

If S satisfies Property (#), then $|K_{C_1\times C_2}|^G$ is spanned by p_g divisors which decompose as a union of fibres for the natural projections $C_1 \times C_2 \to C_i$, $i = 1, 2$. Since two fibres either do not intersect or they intersect transversally at one point, this makes the base locus of $|K_{C_1\times C_2}|^G$ explicit.

Remark 4.2. Observe that Property (#) always holds for G abelian group, and it is possibly satisfied for other non-abelian groups, since we are only interested in those characters of G for which the left-hand side of (4.2) is not zero.

Remark 4.3. In terms of the representation theory, Property (#) translates as

$$\langle \chi^1_{\mathrm{can}}, \chi\rangle \neq 0 \quad \text{and} \quad \langle \chi^2_{\mathrm{can}}, \overline{\chi}\rangle \neq 0 \implies \deg(\chi) = 1$$

for each irreducible character χ, where χ^i_{can} is the character of the canonical representation of C_i, $i = 1, 2$.

Thus, once χ^1_{can} and χ^2_{can} are determined using the Chevalley–Weil formula, verifying whether Property (#) holds reduces to a simple numerical computation.

In Section 4.3, we explain how to compute the self-intersection of the mobile part M of $|K_{C_1\times C_2}|^G$ under the assumption that Property (#) holds.

Note that the difference $M^2 - \widehat{M}^2$ is the sum of the correction terms arising from each isolated base-point of M.

To finish the computation of the degree, whenever $p_g(S) = 3$, we use iteratively for each base point of M the Correction Term formula (Theorem 4.25), which provides the correction term of each base point to the difference $M^2 - \widehat{M}^2$. Such formula is a generalization of the formula presented in [20] and it seems of independent interest, so that it is presented in a more general setting.

Once we have determined both M^2 and $M^2 - \widehat{M}^2$, then the degree of the canonical map of S is obtained by rearranging formula (4.1) as follows:

$$\deg(\Phi_S) = \frac{1}{|G|}\cdot\left(M^2 - (M^2 - \widehat{M}^2)\right).$$

4.1. Base locus of isotypic components of canonical representations of actions on curves

Let C be a Riemann surface, $G < \mathrm{Aut}(C)$ a finite group, $C' := C/G$ its quotient, and $\lambda\colon C \to C'$ the quotient map.

G acts on $H^{1,0}(C)$ via the canonical representation:

$$(g \cdot \omega)_p := (dg^{-1})^*_p \omega_{g^{-1}\cdot p}.$$

Let us denote by χ_{can} the character of the canonical representation, which takes the name of *canonical character*. The canonical representation can be split as a direct sum of irreducible representations:

$$H^{1,0}(C) = \bigoplus_{\chi \in \mathrm{Irr}(G)} H^{1,0}(C)^{\chi}.$$

Here, $H^{1,0}(C)^{\chi}$ is the *isotypic component* of $H^{1,0}(C)$ of character χ. In terms of characters, the above splitting translates as

$$\chi_{\mathrm{can}} = \sum_{\chi \in \mathrm{Irr}(G)} \langle \chi_{\mathrm{can}}, \chi \rangle \cdot \chi.$$

We shall use the algorithm developed in [19] and implemented in the computational algebra system MAGMA to compute the canonical character χ_{can} of any Galois branched covering.

The aim of this section is to investigate the base locus of the associated subsystem $|K_C|^{\chi}$ given by the isotypic component $H^{1,0}(C)^{\chi}$. Let us give first some preliminary results.

Notation. Given a point $q \in C'$, the divisor $\lambda^{-1}(q)$ is considered with the reduced structure.

Lemma 4.4. *Consider a G-invariant subspace $W \subseteq H^{1,0}(C)$. For any $p \in \lambda^{-1}(q)$, $q \in C'$, let $t_p := \min_{\omega \in W} \mathrm{ord}(\omega)$ be the minimal order of a 1-form in W at p. Then, all t_p are equal to the same number, denoted by t_q. Therefore, the base locus of $|W|$ is a union of orbits*

$$Bs\big(|W|\big) = \sum_q t_q \lambda^{-1}(q).$$

Furthermore, there exists a general form $\omega \in W$ with order exactly t_q at each $p \in \lambda^{-1}(q)$.

Proof. For every point $p \in \lambda^{-1}(q)$, there exists a 1-form $\omega_p \in W$ with order t_p at p, by the definition of t_p. Given $g \in G$, then $g \cdot \omega_p$ belongs to the invariant subspace W too, and it vanishes at $g \cdot p$ with multiplicity t_p, so that $t_{g\cdot p} \le t_p$. Hence, all t_p are equal to the same number, denoted as t_q.

We observe that a generic linear combination ω of the $|\lambda^{-1}(q)|$ 1-forms ω_p obtained in this way has order t_q at each point of $\lambda^{-1}(q)$. ■

REMARK 4.5. Let $\omega \in W$ be a 1-form of Lemma 4.4, with order t_q at each point $p \in \lambda^{-1}(q)$. Given $g \in G$, then $g \cdot \omega \in W$ is a 1-form with order t_q at each point $p \in \lambda^{-1}(q)$.

LEMMA 4.6. *Let $f \in \mathcal{M}(C/G) = \mathcal{M}(C)^G$ be a non-zero invariant meromorphic function. Denote by $H^{1,0}(C)^\chi_f$ the subspace of $H^{1,0}(C)^\chi$ consisting of forms ω such that $f\omega$ is a holomorphic form. Then,*

$$f\colon H^{1,0}(C)^\chi_f \to f \cdot H^{1,0}(C)^\chi_f \subseteq H^{1,0}(C), \quad \omega \mapsto f\omega \tag{4.3}$$

is a G-equivariant isomorphism. In particular, $f \cdot H^{1,0}(C)^\chi_f$ is a G-invariant subspace of $H^{1,0}(C)^\chi$.

PROOF. $H^{1,0}(C)^\chi_f$ is G-invariant: given $g \in G$ and $\omega \in H^{1,0}(C)^\chi_f$, then $f(g \cdot \omega) = g \cdot (f\omega)$ is holomorphic since f is G-invariant, and $f\omega$ is holomorphic. This shows immediately that the map of (4.3) is G-equivariant. From the Schur lemma, then the image of (4.3) is contained in $H^{1,0}(C)^\chi$. However, f is not the zero function, so (4.3) is injective. ■

DEFINITION 4.7. Let X be a Riemann surface and $q \in X$. Let us define

$$k_q := \min\left\{m \in \mathbb{N} : h^0(X, mq) \geq 2\right\}$$

as the *minimal non-gap of q*. k_q is therefore the smallest number such that X admits a non-constant meromorphic function f with only one pole at q, of order k_q.

REMARK 4.8. From the Riemann–Roch theorem, we have

$$h^0\big(X, (g(X)+1)q\big) = h^0\big(X, K - \big(g(X)+1\big)q\big) + 2 \geq 2.$$

Therefore,

$$k_q \leq g(X) + 1.$$

In other words, k_q is the minimum of the complement of the set of the Weierstrass gaps for q. In particular, $k_q = g(X) + 1$ if q is not a Weierstrass point; otherwise, $k_q < g(X) + 1$.

Lemma 4.4 applies to $H^{1,0}(C)^\chi$, so the base locus of $|K_C|^\chi$ is

$$Bs\big(|K_C|^\chi\big) = \sum_q t^\chi_q \lambda^{-1}(q),$$

for some positive integers t^χ_q, which we still need to determine.

We denote by ρ_χ an irreducible representation of G of character χ.

We have the following lemma.

LEMMA 4.9. *Let us fix a point $q \in C/G$ of ramification index m_q. Let h be the local monodromy of a point $p \in \lambda^{-1}(q)$; hence, $o(h) = m_q$. There exists*

$$a_q^\chi \in \left\{j \in [0, \dots, m_q - 1] \colon e^{\frac{2\pi i}{m_q} j} \in \operatorname{Spectrum}\big(\rho_\chi(h)\big)\right\}$$

and a non-negative integer $0 \leq k_q^\chi < k_q \leq g(C/G) + 1$ such that

$$t_q^\chi = m_q - a_q^\chi - 1 + k_q^\chi m_q,$$

where k_q is the minimal non-gap of q.

The values a_q^χ and k_q^χ depend only on q and χ and not on the choice of $p \in \lambda^{-1}(q)$.

PROOF. We observe that the action of h on $H^{1,0}(C)^\chi$ is diagonalizable, and its spectrum is contained in the set of the m_q-roots of the unity. Hence, the action of h decomposes $H^{1,0}(C)^\chi$ as

$$H^{1,0}(C)^\chi = \bigoplus_{j=0}^{m_q-1} V_j,$$

where V_j is the eigenspace of eigenvalue ξ^j, and ξ is the first m_q-root of the unity (V_j may be zero, whenever ξ^j is not an eigenvalue of h).

Let $\omega_j \in V_j$ be an eigenvector. We determine the order of ω_j at the point p. By definition of local monodromy, there exists a local coordinate z such that the action of h in a neighborhood of p is $z \mapsto \xi z$. We write $\omega_j = f(z)dz$ locally around this neighborhood of p. We get

$$\xi^j f(z)dz = h \cdot \big(f(z)dz\big) = (h^{-1})^* \big(f(z)dz\big) = f(\xi^{m_q-1}z)\xi^{m_q-1}dz.$$

Hence, f satisfies $f(\xi^{m_q-1}z) = \xi^{j+1} f(z)$, forcing it to be $f(z) = z^{m_q-j-1} g(z^{m_q})$, for some holomorphic function g. Hence, $\operatorname{ord}_p(\omega_j)$ is congruent to $m_q - j - 1$ modulo m_q.

Applying Lemma 4.4 to $W = H^{1,0}(C)^\chi$, we find a form $\omega \in H^{1,0}(C)^\chi$ with order t_q^χ at each point of $\lambda^{-1}(q)$. Let us write ω as a $\omega = \sum_{j=0}^{m_q-1} \omega_j$, with $\omega_j \in V_j$. Since each ω_j has different order at p, then

$$t_q^\chi = \operatorname{ord}_p(\omega) = \min_{\omega_j \neq 0} \left\{\operatorname{ord}_p(\omega_j)\right\}.$$

In other words, there exists $j_0 \in [0, \dots, m_q - 1]$ such that $t_q^\chi = \operatorname{ord}_p(\omega_{j_0})$.

Since ω_{j_0} is an eigenvector of eigenvalue ξ^{j_0}, then $t_q^\chi = \operatorname{ord}_p(\omega_{j_0})$ is congruent to $m_q - j_0 - 1$ modulo m_q; let us say $t_q^\chi = m_q - j_0 - 1 + k_{j_0} m_q$, for some non-negative integer k_{j_0}.

We claim that $k_{j_0} < k_q$, k_q being the minimal non-gap of $q \in C/G$. By contradiction, if $k_{j_0} \geq k_q$, then we use the definition of k_q to pick up a meromorphic function $f \in \mathcal{M}(C/G) = \mathcal{M}(C)^G$ with only one pole at q of order $\operatorname{ord}_q(f) = -k_q$. In this case, then $f\omega$ is a holomorphic form. Indeed, by definition of f, the only poles of $f\omega$ that may occur lie on $\lambda^{-1}(q)$, but the order of $f\omega$ at each $g \cdot p \in \lambda^{-1}(q)$ is

$$\begin{aligned}\operatorname{ord}_{g\cdot p}(f\omega) &= \operatorname{ord}_{g\cdot p}(\omega) + \operatorname{ord}_{g\cdot p}(f) = t_q^\chi - k_q m_q \\ &= m_q - j_0 - 1 + (k_{j_0} - k_q)m_q \geq 0.\end{aligned}$$

Furthermore, from Lemma 4.6, then $f\omega \in H^{1,0}(C)^\chi$. However, this would contradict the definition of t_q^χ since $\operatorname{ord}_p(f\omega) = t_q^\chi - k_q m_q < t_q^\chi$.

To summarize, we have proved

$$t_q^\chi = m_q - j_0 - 1 + k_{j_0} m_q,$$

where j_0 is one of the integers such that $\xi^{j_0} \in \operatorname{Spectrum}(\rho_\chi(h))$, and $k_{j_0} < k_q$.

It is straightforward to see that such integers j_0 and k_{j_0} do not depend on the choice of $p \in \lambda^{-1}(q)$. ■

Theorem 4.10 (Base locus formula). *The base locus of* $|K_C|^\chi$ *is*

$$Bs\big(|K_C|^\chi\big) = \sum_q (m_q - a_q^\chi - 1 + k_q^\chi m_q)\lambda^{-1}(q),$$

where the non-negative integers a_q^χ *and* k_q^χ *are those defined in Lemma 4.9.*

Proof. It suffices to apply Lemma 4.9 to each point $q \in C/G$. ■

Remark 4.11. Under suitable assumptions, it is possible to determine exactly a_q^χ and k_q^χ.

For instance, if $C/G \cong \mathbb{P}^1$, then $k_q = g(C/G) + 1 = 1$, for any $q \in \mathbb{P}^1$. Hence, $k_q^\chi = 0$, and we get

$$t_q^\chi = m_q - a_q^\chi - 1.$$

Moreover, if one of the following holds:

- χ is an irreducible character of degree 1, or
- the local monodromy h is in the center of G,

then $\rho_\chi(h) = \frac{\chi(h)}{\chi(1)} \cdot \mathrm{Id}$ is a multiple of the identity.

This is obvious when the character has degree one. Instead, when the local monodromy is central, this is a result we take from [14].

Under one of these two conditions, then $a_q^\chi \in [0, \dots, m_q - 1]$ is the only integer such that $\chi(h) = e^{\frac{2\pi i}{m_q} a_q^\chi}\chi(1)$.

We deduce then the following immediate consequence from Theorem 4.10 and Remark 4.11.

COROLLARY 4.12. *Assume $C/G \cong \mathbb{P}^1$, and χ is an irreducible character of degree* 1. *Then,*

$$Bs\big(|K_C|^\chi\big) = \sum_q (m_q - a_q^\chi - 1)\lambda^{-1}(q),$$

where $a_q^\chi \in [0, \ldots, m_q - 1]$ is the only non-negative integer such that $\chi(h) = e^{\frac{2\pi i}{m_q} a_q^\chi}$, with h local monodromy of a point p over q.

4.2. *The canonical system of a product-quotient surface*

Let us consider a product-quotient surface S given by a pair of curves C_1 and C_2 and a finite group G acting (faithfully) on both of them. Let $X := (C_1 \times C_2)/G$ be the quotient model of S.

According to the previous section, then G induces the canonical representation on $H^{1,0}(C_i)$; let χ_{can}^i be their canonical characters, respectively, $i = 1, 2$.

THEOREM 4.13. *Every G-invariant global holomorphic 2-form of $C_1 \times C_2$ extends uniquely to a global holomorphic 2-form on the minimal resolution of the singularities $\rho\colon S \to X$ of X. It holds that*

$$H^{2,0}(S) = H^{2,0}(C_1 \times C_2)^G = \bigoplus_{\chi \in \mathrm{Irr}(G)} \big(H^{1,0}(C_1)^\chi \otimes H^{1,0}(C_2)^{\overline{\chi}}\big)^G. \tag{4.4}$$

Furthermore,

$$p_g(S) = \sum_{\chi \in \mathrm{Irr}(G)} \langle \chi_{\mathrm{can}}^1, \chi \rangle \cdot \langle \chi_{\mathrm{can}}^2, \overline{\chi} \rangle.$$

PROOF. Denote by X° the smooth locus of X, i.e. the codimension 2 locus consisting of the image of those points of $C_1 \times C_2$ with a trivial stabilizer. Each global holomorphic 2-form of X° extends uniquely to a global holomorphic 2-form of $C_1 \times C_2$, via the pullback map $\lambda_{12}^*\colon H^{2,0}(X^\circ) \to H^{2,0}(C_1 \times C_2)$, resulting in a monomorphism onto the invariant subspace $H^{2,0}(C_1 \times C_2)^G$. On the other hand, the minimal resolution of the singularities $\rho\colon S \to X$ is an isomorphism on X°; hence, $(\rho^{-1})^*\colon H^{2,0}(S) \to H^{2,0}(X^\circ)$ is a monomorphism. Furthermore, each global holomorphic 2-form on the smooth locus X° of X extends uniquely to a global holomorphic 2-form on S, by Freitag's theorem [21, Satz 1], so $(\rho^{-1})^*$ is an epimorphism too.

Thus $H^{2,0}(S)$ is sent isomorphically via $\lambda_{12}^* \circ (\rho^{-1})^*$ onto the invariant subspace $H^{2,0}(C_1 \times C_2)^G \subseteq H^{2,0}(C_1 \times C_2)$. Finally, by applying Künneth formula and writing $H^{1,0}(C_i)$ as the direct sum of isotypic components, we get

$$H^{2,0}(C_1 \times C_2)^G = \bigoplus_{\chi, \eta \in \mathrm{Irr}(G)} \big(H^{1,0}(C_1)^\chi \otimes H^{1,0}(C_2)^\eta\big)^G.$$

Formula (4.4) follows just from the Schur lemma. Indeed, the dimension of any piece of the sum is $\langle \chi^1_{\mathrm{can}}, \chi\rangle \cdot \langle \chi^2_{\mathrm{can}}, \eta\rangle \cdot \langle \chi\eta, 1\rangle$. However, $\langle \chi\eta, 1\rangle = \langle \chi, \overline{\eta}\rangle$, which is equal to 1 only for $\eta = \overline{\chi}$, and 0 otherwise. ■

Remark 4.14. Using an analogous proof such as that of Theorem 4.13, one can say in general that

$$H^{i,0}(S) = H^{i,0}(C_1 \times C_2)^G$$

by Freitag's theorem [21, Satz 1]. Hence, another immediate consequence firstly observed by Serrano in [33, Prop. 2.2] is a formula for the irregularity of S:

$$q(S) = g(C_1/G) + g(C_2/G).$$

In particular, S is regular if and only if $C_i/G \cong \mathbb{P}^1$.

Let us recall the following classical lemma of representation theory.

Lemma 4.15. *Let us consider an irreducible representation $\phi_\chi\colon G \to \mathrm{GL}(V)$ afforded by a character χ, of degree $n := \chi(1)$. Consider a basis $v_1, \dots, v_n$ of V and its dual basis $e_1, \dots, e_n$ of V^*. Then, $(V \otimes V^*)^G$ is one-dimensional and it is generated by $v_1 \otimes e_1 + \dots + v_n \otimes e_n$.*

We use the previous lemma to describe a basis of $(H^{1,0}(C_1)^\chi \otimes H^{1,0}(C_2)^{\overline{\chi}})^G$.

Remark 4.16. Let us consider an irreducible representation $\phi_\chi\colon G \to \mathrm{GL}(V)$ of character χ. Let $n := \chi(1)$ be the degree of ϕ_χ. Then, $H^{1,0}(C_1)^\chi \otimes H^{1,0}(C_2)^{\overline{\chi}}$ is the direct sum of a certain number of copies of $V \otimes V^*$ (the exact number of copies is $\langle \chi^1_{\mathrm{can}}, \chi\rangle \cdot \langle \chi^2_{\mathrm{can}}, \overline{\chi}\rangle$). Consequently, its invariant subspace $(H^{1,0}(C_1)^\chi \otimes H^{1,0}(C_2)^{\overline{\chi}})^G$ is a direct sum of the same number of copies of the invariant subspace $(V \otimes V^*)^G$. Let us fix a basis $\{\omega_1, \dots, \omega_n\}$ of V and the (dual) basis $\{\eta_1, \dots, \eta_n\}$ on V^*. We denote by $\{\omega^k_1, \dots, \omega^k_n\}$ the corresponding basis of the k-th copy of V in $H^{1,0}(C_1)^\chi$, $k = 1, \dots, \langle \chi^1_{\mathrm{can}}, \chi\rangle$, and by $\{\eta^l_1, \dots, \eta^l_n\}$ the corresponding basis of the l-th copy of V^* in $H^{1,0}(C_2)^{\overline{\chi}}$, $l = 1, \dots, \langle \chi^2_{\mathrm{can}}, \overline{\chi}\rangle$. Lemma 4.15 applies for any copy of $(V \otimes V^*)^G$, so that

$$\big(H^{1,0}(C_1)^\chi \otimes H^{1,0}(C_2)^{\overline{\chi}}\big)^G = \bigoplus_{k,l} \langle \omega^k_1 \otimes \eta^l_1 + \dots + \omega^k_n \otimes \eta^l_n\rangle. \tag{4.5}$$

Definition 4.17. We denote by $|K_{C_1\times C_2}|^G$ the linear subsystem of the canonical system of $C_1 \times C_2$ given by the subspace of invariant 2-forms of $C_1 \times C_2$.

We give a theoretical description of the canonical map Φ_{K_S} of S. From Theorem 4.13, the (rational) map $\Phi_{K_S} \circ \lambda_{12}$ is induced by the linear subsystem $|K_{C_1\times C_2}|^G$.

The situation is the following:

$$
\begin{tikzcd}
 & X & & \\
C_1 \times C_2 \arrow[ur, "\lambda_{12}"] \arrow[rr, dashed, "\lambda_{12}"] \arrow[d] \arrow[drr, "\Phi_{K_{C_1\times C_2}}"'] \arrow[rrr, dashed, bend right, "\Phi_{|K_{C_1\times C_2}|^G}"] & & S \arrow[ul, "\rho"'] \arrow[r, dashed, "\Phi_{K_S}"] & \mathbb{P}^{p_g-1} \\
\mathbb{P}^{g_1-1} \times \mathbb{P}^{g_2-1} \arrow[rr, "\text{Segre}"] & & \mathbb{P}^{g_1 g_2-1}. \arrow[ur, dashed, "proj"'] &
\end{tikzcd}
$$

Let us fix a basis of $H^{1,0}(C_1)$ and $H^{1,0}(C_2)$. Then, $\Phi_{K_S} \circ \lambda_{12}$ is the composition of the product of the canonical maps of C_1 and C_2 with the Segre embedding in $\mathbb{P}^{g_1 g_2-1}$, together with the projection map *proj*. This latter map sends a basis of 2-forms of $C_1 \times C_2$ to a basis of invariant 2-forms defining Φ_{K_S}.

We can use Remark 4.16 to give an explicit description of *proj*, which is defined in coordinates as follows.

Let us fix coordinates ${}^{\chi}x_{ij}^{kl}$ on $\mathbb{P}^{g_1 g_2-1}$, with $1 \le i, j \le \chi(1)$, and $1 \le k \le \langle \chi^1_{\text{can}}, \chi\rangle$, $1 \le l \le \langle \chi^2_{\text{can}}, \overline{\chi}\rangle$. Then,

$$
proj\big(({}^{\chi}x_{ij}^{kl} : \chi, i, j, k, l)\big) = \big({}^{\chi}x_{11}^{kl} + \cdots + {}^{\chi}x_{nn}^{kl} : \chi \in \operatorname{Irr}(G),\ n = \chi(1), k, l\big).
$$

4.3. *Base locus of the invariant subsystem* $|K_{C_1\times C_2}|^G$

Given an irreducible character $\chi \in \operatorname{Irr}(G)$, we have the following series of inclusions:

$$
\big(H^{1,0}(C_1)^{\chi} \otimes H^{1,0}(C_2)^{\overline{\chi}}\big)^G \subseteq H^{1,0}(C_1)^{\chi} \otimes H^{1,0}(C_2)^{\overline{\chi}} \subseteq H^{2,0}(C_1 \times C_2).
$$

Let us define the associated subsystems of $|K_{C_1\times C_2}|$ given by these subspaces.

Definition 4.18. We denote by $|K_{C_1}|^{\chi} \otimes |K_{C_2}|^{\overline{\chi}}$ and by $(|K_{C_1}|^{\chi} \otimes |K_{C_2}|^{\overline{\chi}})^G$ the associated subsystems of the canonical linear system of $C_1 \times C_2$ given by $H^{1,0}(C_1)^{\chi} \otimes H^{1,0}(C_2)^{\overline{\chi}}$ and $(H^{1,0}(C_1)^{\chi} \otimes H^{1,0}(C_2)^{\overline{\chi}})^G$, respectively.

Theorem 4.13 permits us to describe the base locus of $|K_{C_1\times C_2}|^G$ in terms of the base locus of its pieces $(|K_{C_1}|^{\chi} \otimes |K_{C_2}|^{\overline{\chi}})^G$, $\chi \in \operatorname{Irr}(G)$. More precisely, we have

$$
Bs\big(|K_{C_1\times C_2}|^G\big) = \bigcap_{\langle \chi^1_{\text{can}}, \chi\rangle \neq 0,\ \langle \chi^2_{\text{can}}, \overline{\chi}\rangle \neq 0} Bs\big((|K_{C_1}|^{\chi} \otimes |K_{C_2}|^{\overline{\chi}})^G\big). \tag{4.6}
$$

Notation. We denote $\lambda\colon X \to C_1/G \times C_2/G$, and $\lambda_i\colon C_i \to C_i/G$, $i = 1, 2$. Furthermore, let us denote

$$
B_q^{\text{vert}} := \{q\} \times C_2/G, \quad \text{and} \quad B_l^{\text{hor}} := C_1/G \times \{l\},
$$

where $q \in C_1/G$ and $l \in C_2/G$. Similarly, R_q^{vert} and R_l^{hor} refer to the reduced inverse images on $C_1 \times C_2$ of B_q^{vert} and B_l^{hor}:

$$R_q^{\mathrm{vert}} := \frac{1}{m_q}(\lambda \circ \lambda_{12})^*\big(\{q\} \times C_2/G\big), \quad R_l^{\mathrm{hor}} := \frac{1}{m_l}(\lambda \circ \lambda_{12})^*\big(C_1/G \times \{l\}\big).$$

REMARK 4.19. With this notation, then the branch locus of $\lambda \circ \lambda_{12}\colon C_1 \times C_2 \to C_1/G \times C_2/G$ is the grid given by the union of B_q^{vert} and B_l^{hor} with $q \in \mathrm{Crit}(\lambda_1)$ and $l \in \mathrm{Crit}(\lambda_2)$.

Base Locus formula theorem 4.10 provides a formula for the base locus of $|K_{C_1}|^\chi \otimes |K_{C_2}|^{\overline{\chi}}$.

THEOREM 4.20. *The (schematic) base locus of the linear subsystem $|K_{C_1}|^\chi \otimes |K_{C_2}|^{\overline{\chi}}$ of $|K_{C_1 \times C_2}|$ is pure in codimension 1 and is equal to*

$$Bs\big(|K_{C_1}|^\chi \otimes |K_{C_2}|^{\overline{\chi}}\big) = \sum_{q \in \mathrm{Crit}(\lambda_1)} t_q^\chi R_q^{\mathrm{vert}} + \sum_{l \in \mathrm{Crit}(\lambda_2)} t_l^{\overline{\chi}} R_l^{\mathrm{hor}}, \tag{4.7}$$

where t_q^χ and $t_l^{\overline{\chi}}$ are the non-negative integers of Lemma 4.9.

COROLLARY 4.21. *Let χ be a character of degree 1. Then,*

$$\big(H^{1,0}(C_1)^\chi \otimes H^{1,0}(C_2)^{\overline{\chi}}\big)^G = H^{1,0}(C_1)^\chi \otimes H^{1,0}(C_2)^{\overline{\chi}}$$

and the base locus of its associated linear subsystem

$$\big(|K_{C_1}|^\chi \otimes |K_{C_2}|^{\overline{\chi}}\big)^G = |K_{C_1}|^\chi \otimes |K_{C_2}|^{\overline{\chi}}$$

is given by the formula (4.7) of Theorem 4.20.

Assume furthermore that $C_i/G \cong \mathbb{P}^1$, for $i = 1, 2$. Then, t_q^χ and $t_l^{\overline{\chi}}$ of (4.7) are the unique non-negative integers with $0 \le t_q^\chi \le m_q - 1$ and $0 \le t_l^{\overline{\chi}} \le m_l - 1$ satisfying

$$\chi(h) = e^{\frac{2\pi i}{m_q}(m_q - t_q^\chi - 1)} \quad \textit{and} \quad \chi(g) = e^{\frac{2\pi i}{m_l}(t_l^{\overline{\chi}} + 1)},$$

where h is the local monodromy of a point over q, and g is the local monodromy of a point over l.

PROOF. The first claim is straightforward since every $v \otimes w \in H^{1,0}(C_1)^\chi \otimes H^{1,0}(C_2)^{\overline{\chi}}$ is G-invariant

$$g \cdot (v \otimes w) = \big(\chi(g)v\big) \otimes \big(\overline{\chi}(g)w\big) = \big|\chi(g)\big|^2 v \otimes w = v \otimes w.$$

The rest of the thesis follows from Remark 4.11 and from the fact that since $t_l^{\overline{\chi}}$ is the unique non-negative integer such that

$$\overline{\chi}(g) = e^{\frac{2\pi i}{m_l}(m_l - t_l^{\overline{\chi}} - 1)},$$

then it is the unique non-negative integer such that $\chi(g) = e^{\frac{2\pi i}{m_l}(t_l^{\overline{\chi}}+1)}$. ■

LEMMA 4.22. *Suppose S satisfies Property (#). Then, the fixed part of the linear system $|K_{C_1\times C_2}|^G$ is*

$$\mathrm{Fix}\big(|K_{C_1\times C_2}|^G\big) = \sum_{q\in \mathrm{Crit}(\lambda_1)} \Big(\min_{\chi : \langle \chi^1_{\mathrm{can}}, \chi\rangle \neq 0, \langle \chi^2_{\mathrm{can}}, \overline{\chi}\rangle \neq 0} t_q^{\chi}\Big) R_q^{\mathrm{vert}} + \sum_{l\in \mathrm{Crit}(\lambda_2)} \Big(\min_{\chi : \langle \chi^1_{\mathrm{can}}, \chi\rangle \neq 0, \langle \chi^2_{\mathrm{can}}, \overline{\chi}\rangle \neq 0} t_l^{\overline{\chi}}\Big) R_l^{\mathrm{hor}}. \tag{4.8}$$

PROOF. The fixed part of $|K_{C_1\times C_2}|^G$ is the common divisor of the fixed parts of those pieces $(|K_{C_1}|^\chi \otimes |K_{C_2}|^{\overline{\chi}})^G$ that are non-empty, for χ irreducible character. By Property (#), then χ is of degree 1; hence, Corollary 4.21 applies and the fixed part of $(|K_{C_1}|^\chi \otimes |K_{C_2}|^{\overline{\chi}})^G$ amounts to

$$\sum_{q\in \mathrm{Crit}(\lambda_1)} t_q^{\chi} R_q^{\mathrm{vert}} + \sum_{l\in \mathrm{Crit}(\lambda_2)} t_l^{\overline{\chi}} R_l^{\mathrm{hor}}.$$

The common divisor of these fixed parts is the right-hand side of (4.8). ■

Let $|M|$ be the moving part of $|K_{C_1\times C_2}|^G$. By the definition of M, then

$$M \equiv K_{C_1\times C_2} - \mathrm{Fix}\big(|K_{C_1\times C_2}|^G\big).$$

Suppose S satisfies Property (#). Thus, $\mathrm{Fix}(|K_{C_1\times C_2}|^G)$ is a union of fibres by equation (4.8). To compute M^2 is then sufficient to know the intersection product of

$$K_{C_1\times C_2}\cdot R_q^{\mathrm{vert}}, \quad K_{C_1\times C_2}\cdot R_l^{\mathrm{hor}}, \quad (R_q^{\mathrm{vert}})^2, \quad (R_l^{\mathrm{hor}})^2, \quad R_q^{\mathrm{vert}}\cdot R_l^{\mathrm{hor}}.$$

We compute them.

R_q^{vert} can be written as the sum of $|G|/m_q$ components $\{g\cdot p\}\times C_2$, with p point over q, and $g\in G$. $\{g\cdot p\}\times C_2$ has self-intersection zero (since two points are always homologous on a connected variety, and then the fibres of $C_1\times C_2 \to C_1$ are always numerically equivalent). Thus, we can use the genus formula to get

$$K_{C_1\times C_2}\cdot \big(\{g\cdot p\}\times C_2\big) = 2g(C_2) - 2 - \big(\{g\cdot p\}\times C_2\big)^2 = 2g(C_2) - 2.$$

The same reasoning works for a horizontal divisor R_l^{hor}. Thus, we have

$$K_{C_1\times C_2}\cdot R_q^{\mathrm{vert}} = \frac{|G|}{m_q}\big(2g(C_2)-2\big), \quad K_{C_1\times C_2}\cdot R_l^{\mathrm{hor}} = \frac{|G|}{m_l}\big(2g(C_1)-2\big).$$

Analogously,

$$(R_q^{\text{vert}})^2 = (R_l^{\text{hor}})^2 = 0, \quad \text{and} \quad R_q^{\text{vert}} \cdot R_l^{\text{hor}} = \frac{|G|^2}{m_q m_l}.$$

4.4. A formula for the degree of the canonical map

In the previous subsection, we have seen that the rational map $\Phi_{K_S} \circ \lambda_{12}$ is induced by the linear subsystem $|K_{C_1 \times C_2}|^G$, which is generated by p_g invariant 2-forms defining Φ_{K_S}:

$$\begin{tikzcd} C_1 \times C_2 \arrow[r, dashed, "\lambda_{12}"] \arrow[rr, dashed, bend right, "\Phi_{|K_{C_1\times C_2}|^G}"'] & S \arrow[r, dashed, "\Phi_{K_S}"] & \mathbb{P}^{p_g - 1}. \end{tikzcd}$$

We resolve the indeterminacy of $\Phi_{|K_{C_1\times C_2}|^G} = \Phi_{K_S} \circ \lambda_{12}$ by a sequence of blowups:

$$\begin{tikzcd} \widehat{C_1 \times C_2} \arrow[r] \arrow[dr, "\Phi_{\widehat{M}}"'] & C_1 \times C_2 \arrow[d, dashed, "\Phi_{|K_{C_1\times C_2}|^G}"] \\ & \mathbb{P}^{p_g - 1}. \end{tikzcd}$$

LEMMA 4.23. *The map Φ_{K_S} is not composed with a pencil if and only if $\widehat{M}^2$ is positive.*

PROOF. The map Φ_{K_S} is composed with a pencil if and only if $\Phi_{\widehat{M}}$ is composed with a pencil. The image Σ of $\Phi_{\widehat{M}}$ is a curve if and only if we are able to pick up two general hyperplanes H_1 and H_2 of $\mathbb{P}^{p_g-1}$ such that $H^2_{|\Sigma} = H_1 \cdot H_2 \cdot \Sigma = 0$. However, $\widehat{M} = \Phi^*_{\widehat{M}}(H)$; hence, $H^2_{|\Sigma}$ is zero if and only if $\widehat{M}^2$ is equal to zero. ∎

Let us suppose $\widehat{M}^2 > 0$, so that Φ_{K_S} has image Σ of dimension 2. In this case, then $\Phi_{\widehat{M}}$ is a finite morphism, and by projection formula,

$$\widehat{M}^2 = \deg(\Phi_{\widehat{M}}) \deg(\Sigma) = \deg(\Phi_{K_S}) \deg(\Sigma)|G|,$$

which gives formula (4.1).

4.5. The correction term to the self-intersection of a 2-dimensional linear system with only isolated base points

As remarked in the introduction of this section, $M^2 - \widehat{M}^2$ is the sum of the correction terms arising from each isolated base-point of M, the mobile part of the linear subsystem $|K_{C_1 \times C_2}|^G$.

The contribution to the correction term of any isolated base-point may be easily computed whenever S satisfies Property (#).

Let us fix a base-point $(p_1, p_2) \in C_1 \times C_2$ of the mobile part M. The point p_1 is over $q \in C_1/G$ and p_2 is over $l \in C_2/G$. Let us fix an irreducible character χ. We can always choose a basis of $H^{1,0}(C_1)^\chi$ such that each 1-form of the basis has the minimum order t_q^χ at p_1, which is the positive integer computed in Lemma 4.9.

Similarly, we can choose a basis of $H^{1,0}(C_2)^{\overline{\chi}}$ such that each 1-form of the basis has minimum order $t_l^{\overline{\chi}}$ at p_2. The choice of this pair of bases gives via tensor product a natural basis of $H^{1,0}(C_1)^\chi \otimes H^{1,0}(C_2)^{\overline{\chi}}$, which is a G-invariant subspace since Property (#) holds; namely, χ is of degree one. This permits us to conclude that the divisors spanning the linear subsystem $|K_{C_1 \times C_2}|^G$ can be written in a neighborhood of (p_1, p_2) as

$$t_q^\chi R_q^{\mathrm{vert}} + t_l^{\overline{\chi}} R_l^{\mathrm{hor}}, \qquad \chi \text{ such that } \langle \chi_{\mathrm{can}}^1, \chi \rangle \neq 0,\ \langle \chi_{\mathrm{can}}^2, \overline{\chi} \rangle \neq 0.$$

Finally, it is sufficient to remove the fixed part of $|K_{C_1 \times C_2}|^G$ computed in Lemma 4.22 to get how the divisors spanning M are written in a neighborhood of (p_1, p_2). So, the linear system M is spanned by $p_g(S)$ divisors locally near (p_1, p_2) of the form

$$a_1 R_q^{\mathrm{vert}} + b_1 R_l^{\mathrm{hor}}, \quad \ldots, \quad a_{p_g} R_q^{\mathrm{vert}} + b_{p_g} R_l^{\mathrm{hor}}.$$

Since we assumed that (p_1, p_2) is a base-point and M has no fixed components, then without loss of generality, $a_1 = b_2 = 0$.

Note that R_q^{vert} and R_l^{hor} are smooth and intersect transversally at (p_1, p_2).

We provide a formula to directly compute the contribution of (p_1, p_2) to the correction term $M^2 - \widehat{M}^2$ whenever $p_g(S)$ is equal to three. This formula, presented in a slightly more general setting, is a stronger version of [18, Lem. 2]. We recall the following definition.

Definition 4.24. Let M be a (not necessarily complete) linear system on a surface S. The *strict transform* $\widehat{M}$ of M at p is defined as follows. We blow up the basepoint p, take the pullback of the moving part of M, and remove the fixed part of this new linear system. If an infinitely near point of p is a base-point for this linear system, then repeat the procedure until we obtain a (not necessarily complete) linear system $\widehat{M}$ such that no infinitely near point of p is a base point of $\widehat{M}$.

Theorem 4.25 (Correction term formula). *Let M be a two-dimensional linear system on a surface S spanned by D_1, D_2, and D_3. Assume that M has only isolated base-points, smooth for S, and that in a neighborhood of a basepoint p, we can write the divisors D_i as*

$$D_1 = aH, \quad D_2 = bK, \quad \textit{and} \quad D_3 = cH + dK.$$

Here, H and K are reduced, smooth, and intersecting transversally at p and a, b, c, d are non-negative integers, $b \leq a$. Let $\widehat{M}$ be the strict transform of M at p. Then,

$$M^2 - \widehat{M}^2 = \min\{ab,\ ad + bc\}.$$

Proof. The proof follows from Lemmas 4.26 and 4.27 below. ∎

Lemma 4.26. *Assume that $bc + ad \geq ab$. Then, $M^2 - \widehat{M}^2 = ab$.*

Proof. We prove the lemma by induction on (a, b), with $b \leq a$. Here, we are considering the lexicographic order $\leq$ defined on the lower half plane $\Delta^{\geq} := \{(a, b) : a \geq b\} \subseteq \mathbb{N} \times \mathbb{N}$ as follows:

$$(a', b') \leq (a, b) \quad \text{if and only if } a' < a \text{ or } a' = a \text{ and } b' \leq b.$$

In this case, $\Delta^{\geq}$ admits the *well-ordering principle* and so it holds the *mathematical induction*.

Suppose that $(a, b) = 0$. Then, M is base-point-free and so $\widehat{M}^2 = M^2 = M^2 - ab$. Now suppose that the statement is true for $(a', b') < (a, b)$. We aim to prove it for (a, b). We blow up the base-point p, take the pullback of the divisors D_i, and remove the fixed part, which is the exceptional divisor bE of the blowup. In fact, the pullback of D_3 contains $c + d$ times E and $c + d \geq b$, thanks to $b \leq a$ and to the assumption $bc + ad \geq ab$:

$$a(c + d) \geq bc + ad \geq ab, \quad \text{so} \quad c + d \geq b.$$

Restricted to the preimage of our neighborhood of p, these divisors are

$$a\widehat{H} + (a - b)E, \quad b\widehat{K}, \quad \text{and} \quad c\widehat{H} + d\widehat{K} + (c + d - b)E.$$

Here, $\widehat{H}$ and $\widehat{K}$ are the strict transforms of H and K. Let $\widehat{M}$ be the linear system generated by these three divisors, and then $\widehat{M}^2 = M^2 - b^2$. If $a = b$ or $b = 0$, then $\widehat{M}$ is base-point-free and we are done. Otherwise, on the preimage, the linear system $\widehat{M}$ has precisely one new base-point: the intersection point of $\widehat{K}$ and E. Locally near this point the three divisors spanning $\widehat{M}$ are

$$(a - b)E, \quad b\widehat{K}, \quad \text{and} \quad d\widehat{K} + (c + d - b)E.$$

We need to distinguish two cases, when $(a - b) < b$ or when $(a - b) \geq b$. In the first case $(a - b) < b$, we get $(b, a - b) < (a, b)$. We define new coefficients $a' := b$, $b' := a - b$, $c' := d$, and $d' := c + d - b$. Otherwise, if $(a - b) \geq b$, then $(a - b, b) < (a, b)$, and we define $a' := a - b$, $b' := b$, $c' := c + d - b$, and $d' := d$. For both cases, the new coefficients fulfill the inductive hypothesis.

Thanks to $bc + ad \geq ab$, we have

$$b'c' + a'd' = (a - b)d + b(c + d - b) = ad + bc - b^2 \geq ab - b^2 = (a - b)b = a'b'.$$

By induction, the self-intersection of the new linear system $\widehat{M}$ is equal to

$$\widehat{M}^2 = (M^2 - b^2) - b(a-b) = M^2 - ab. \qquad \blacksquare$$

LEMMA 4.27. *Assume that $bc + ad \leq ab$. Then, $M^2 - \widehat{M}^2 = ad + bc$.*

PROOF. We prove the lemma by induction, once more on (a, b), with $b \leq a$. Thus, we consider the lexicographic order $\leq$ on $\Delta^{\geq}$, as we have done in the proof of Lemma 4.26.

Suppose that $(a, b) = 0$. Then, M is base-point-free and so

$$\widehat{M} = M^2 = M^2 - (0d + 0c).$$

Now suppose that the statement is true for $(a', b') < (a, b)$. Our aim is to prove it for (a, b). We blow up the base-point p, take the pullback of the divisors D_i, and remove the fixed part, which is the exceptional divisor $(c+d)E$ of the blowup, if $c + d \leq b$, or the divisor bE, otherwise. Hence, we need to distinguish two cases.

Let us suppose first that $c + d \leq b$ ($\leq a$). Restricted to the preimage of our neighborhood of p, the divisors are

$$a\widehat{H} + \big(a - (c+d)\big)E, \quad b\widehat{K} + \big(b - (c+d)\big)E, \quad \text{and} \quad c\widehat{H} + d\widehat{K}.$$

Here, $\widehat{H}$ and $\widehat{K}$ are the strict transforms of H and K. Let $\widehat{M}$ be the linear system generated by these three divisors, and then $\widehat{M}^2 = M^2 - (c+d)^2$. On the preimage, the linear system $\widehat{M}$ has at most two new base-points: the intersection points of $\widehat{H}$ and $\widehat{K}$ with E. Locally near these points the three divisors spanning $\widehat{M}$ are respectively

$$a\widehat{H} + \big(a - (c+d)\big)E, \quad \big(b - (c+d)\big)E \quad \text{and} \quad c\widehat{H},$$

and

$$\big(a - (c+d)\big)E, \quad b\widehat{K} + \big(b - (c+d)\big)E \quad \text{and} \quad d\widehat{K}.$$

We claim that for both points the coefficients of these three divisors satisfy the assumption of Lemma 4.26.

Let us verify it for the first point $\widehat{H} \cap E$: if $c \geq (b - (c+d))$, then define $a' := c$, $b' := b - (c+d)$, $c' := a$, and $d' := a - (c+d)$; otherwise, define $a' := b - (c+d)$, $b' := c$, $c' := a - (c+d)$, and $d' := a$. For both the cases, $d' \geq b'$ so that $b'c' + a'd' \geq a'd' \geq a'b'$.

An analogous argument holds for the point $\widehat{K} \cap E$, so Lemma 4.26 applies for both points and the self-intersection of the new linear system $\widehat{M}$ at the final step is

$$\widehat{M}^2 = \big(M^2 - (c+d)^2\big) - \big(b - (c+d)\big)c - \big(a - (c+d)\big)d = M^2 - (ad + bc).$$

It remains to discuss the case $c + d \geq b$.

Take the pullback of the divisors D_i, and remove the fixed part, which this time is the exceptional divisor bE of the blowup. Restricted to the preimage of our neighborhood of p, the divisors D_i are

$$a\widehat{H} + (a-b)E, \quad b\widehat{K}, \quad \text{and} \quad c\widehat{H} + d\widehat{K} + (c+d-b)E.$$

Here, $\widehat{M}^2 = M^2 - b^2$. If $b=0$ or $a=b$, then $\widehat{M}$ is base-point-free. In the first case $b=0$, we get $ad = bc + ad \leq ab = 0$, so $\widehat{M}^2 = M^2 - b^2 = M^2 = M^2 - (ad + bc)$, and we are done. In the second case $a = b$, we get, thanks to the assumptions $ad + bc \leq ab$ and $b \leq c + d$, that $c + d = b = a$, and we are done:

$$\widehat{M}^2 = M^2 - b^2 = M^2 - (ad + bc).$$

It remains to consider when $a - b = 0$ or $b = 0$ does not hold. In this case, on the preimage, the linear system $\widehat{M}$ would have precisely one new base-point, the intersection point of $\widehat{K}$ and E. Locally near this point the three divisors spanning $\widehat{M}$ are

$$(a-b)E, \quad b\widehat{K}, \quad \text{and} \quad d\widehat{K} + (c+d-b)E.$$

We need to distinguish two cases, when $(a-b) < b$ or when $(a-b) \geq b$. In the first case $(a-b) < b$, we get $(b, a-b) < (a, b)$. We define new coefficients $a' := b$, $b' := a-b$, $c' := d$, and $d' := c + d - b$. Otherwise, if $(a-b) \geq b$, then $(a-b, b) < (a, b)$, and we define $a' := a - b$, $b' := b$, $c' := c + d - b$, and $d' := d$. For both cases, the new coefficients fulfill the inductive hypothesis.

Thanks to $bc + ad \leq ab$, we have

$$b'c' + a'd' = (a-b)d + b(c+d-b) = ad + bc - b^2 \leq ab - b^2 = (a-b)b = a'b'.$$

By induction, the self-intersection of the new linear system $\widehat{M}$ is equal to

$$\widehat{M}^2 = (M^2 - b^2) - (a'd' + b'c') = M^2 - b^2 - (ad + bc - b^2) = M^2 - (ad + bc). \quad \blacksquare$$

4.6. *Example of the computation of the degree of the canonical map*

In this section, we give an example of how to compute the degree of the canonical map of a regular product-quotient surface of geometric genus three, whenever Property (#) holds. In addition, in this way, we also show the main steps of the MAGMA script for calculating the degree of the canonical map.

Let us consider the family of surfaces no. 1 in [17, Thm. 2.3], which have degree of the canonical map 18.

Surfaces S of no. 1 of [17, Thm. 2.3] can be described by the following pair of spherical systems of generators of the group $G = S_3 \times \mathbb{Z}_3^2$.

	q_1	q_2	q_3
Branch point	$(1:1)$	$(0:1)$	$(-1:1)$
Generator	$g_1 := (\tau, (1,0))$	$g_2 := (\sigma^2, (2,2))$	$g_3 := (\sigma\tau, (0,1))$

	q_1'	q_2'	q_3'	q_4'
Branch point	$(1:1)$	$(0:1)$	$(1:\nu)$	$(-1:1)$
Generator	$h_1 := (\sigma\tau, 0)$	$h_2 := (\sigma, (1,0))$	$h_3 := (\mathrm{Id}, (1,1))$	$h_4 := (\tau, (1,2))$

Here, σ and τ are a rotation (a 3-cycle) and a reflection (a transposition) of the group S_3, respectively. Meanwhile, the points q_j are the branch points of the first G-covering $C_1 \to \mathbb{P}^1$, and the corresponding g_j is the local monodromy of a point over q_j. A similar description holds for the points q_j' and generators h_j of the second G-covering $C_2 \to \mathbb{P}^1$.

Notice that the second covering depends on one parameter ν, with $\nu \neq -1, 1$ since C_2 is smooth.

Consider the three irreducible characters of S_3, that is, the trivial character 1, the character sgn computing the sign of a permutation, and the only 2-dimensional irreducible character
$$\mu := \frac{1}{2}(\chi_{\mathrm{reg}} - sgn - 1),$$
where χ_{reg} is the character of the regular representation of S_3.

Let us also fix a basis e_1, e_2 of $\mathbb{Z}_3^2$ and consider the dual characters ε_1, ε_2 of e_1 and e_2, i.e. the characters defined by
$$\varepsilon_i(ae_1 + be_2) := \zeta_3^{a\delta_{1i} + b\delta_{2i}}, \quad \zeta_3 := e^{\frac{2\pi i}{3}},$$
where δ_{ij} is the Kronecker delta.

We apply the Chevalley–Weil formula [19, Thm. 2.8] to both curves C_1 and C_2 defining S to compute the canonical characters χ^1_{can} and χ^2_{can}, respectively:
$$\begin{aligned}\chi^1_{\mathrm{can}} &= \varepsilon_1^2 \cdot \varepsilon_2^2 + sgn \cdot \varepsilon_1 \cdot \varepsilon_2 + sgn \cdot \varepsilon_2 + sgn \cdot \varepsilon_1 \\ &\quad + \mu \cdot \varepsilon_1 \cdot \varepsilon_2 + \mu \cdot \varepsilon_1^2 \cdot \varepsilon_2 + \mu \cdot \varepsilon_1 \cdot \varepsilon_2^2; \\ \chi^2_{\mathrm{can}} &= sgn \cdot \varepsilon_1^2 \cdot \varepsilon_2 + sgn \cdot \varepsilon_1^2 \cdot \varepsilon_2^2 + sgn \cdot \varepsilon_1 \cdot \varepsilon_2 + sgn \cdot \varepsilon_1 + sgn \cdot \varepsilon_2^2 \\ &\quad + \mu \cdot \varepsilon_1 + \mu \cdot \varepsilon_2 + 2\mu \cdot \varepsilon_2^2 + sgn \cdot \varepsilon_1^2 + \varepsilon_1^2 + \mu \cdot \varepsilon_1^2 + \mu \cdot \varepsilon_1 \cdot \varepsilon_2.\end{aligned}$$

We notice that the irreducible characters χ such that χ occurs on χ^1_{can} and $\overline{\chi}$ occurs on χ^2_{can} have degree one, so Property (#) is satisfied. These characters are precisely
$$sgn \cdot \varepsilon_1 \cdot \varepsilon_2, \quad sgn \cdot \varepsilon_2, \quad \text{and} \quad sgn \cdot \varepsilon_1.$$

From Theorem 4.13, we have that
$$H^{2,0}(S) = \left(H^{1,0}(C_1) \otimes H^{1,0}(C_2)\right)^{S_3 \times \mathbb{Z}_3^2}$$

decomposes into three pieces of dimension one:

$$H^{1,0}(C_1)^{sgn\cdot\varepsilon_1\cdot\varepsilon_2} \otimes H^{1,0}(C_2)^{sgn\cdot\varepsilon_1^2\cdot\varepsilon_2^2}, \quad H^{1,0}(C_1)^{sgn\cdot\varepsilon_2} \otimes H^{1,0}(C_2)^{sgn\cdot\varepsilon_2^2},$$
$$H^{1,0}(C_1)^{sgn\cdot\varepsilon_1} \otimes H^{1,0}(C_2)^{sgn\cdot\varepsilon_1^2}.$$

For each of these three pieces, Corollary 4.21 determines a generator of the associated linear subsystem:

$$R^{\text{vert}}_{(0,1)} + R^{\text{hor}}_{(1,\nu)} + 2R^{\text{hor}}_{(-1,1)}, \quad 2R^{\text{vert}}_{(1,1)} + 2R^{\text{hor}}_{(0,1)}, \quad 2R^{\text{vert}}_{(-1,1)} + 4R^{\text{hor}}_{(-1,1)}.$$

Thus, the above three divisors are spanning the linear system $|K_{C_1\times C_2}|^{S_3\times\mathbb{Z}_3^2}$.

Notice then $|K_{C_1\times C_2}|^{S_3\times\mathbb{Z}_3^2}$ has no fixed part, so that

$$M^2 = \left(2R^{\text{vert}}_{(1,1)} + 2R^{\text{hor}}_{(0,1)}\right)^2 = 4\cdot 2\cdot\left(R^{\text{vert}}_{(1,1)}\cdot R^{\text{hor}}_{(0,1)}\right) = 8\frac{54}{6}\cdot\frac{54}{3} = 24\cdot 54.$$

Furthermore, $|K_{C_1\times C_2}|^{S_3\times\mathbb{Z}_3^2}$ has precisely 81 (non-reduced) isolated base-points $R^{\text{vert}}_{(1,1)} \cap R^{\text{hor}}_{(-1,1)}$. We can compute $M^2 - \widehat{M}^2$ by applying Theorem 4.25, recursively for each base-point of $|K_{C_1\times C_2}|^{S_3\times\mathbb{Z}_3^2}$. Indeed, in a neighborhood of each of these base-points, the three divisors are respectively

$$2R^{\text{hor}}_{(-1,1)}, \quad 2R^{\text{vert}}_{(1,1)}, \quad \text{and} \quad 4R^{\text{hor}}_{(-1,1)},$$

and since $R^{\text{vert}}_{(1,1)}$ and $R^{\text{hor}}_{(-1,1)}$ are transversal, then we are in the situation of Theorem 4.25, with $H = R^{\text{hor}}_{(-1,1)}$ and $K = R^{\text{vert}}_{(1,1)}$, $a = 4$, $b = c = 2$, and $d = 0$. This implies $ad + bc = 4 \le ab = 8$. The correction term is $ab + cd = 4$ for each of the 81 base-points. Thus,

$$M^2 - \widehat{M}^2 = 4\cdot 81.$$

The degree of the canonical map is therefore given by

$$\deg(\Phi_{K_S}) = \frac{1}{54}\widehat{M}^2 = \frac{1}{54}\left(M^2 - (M^2 - \widehat{M}^2)\right) = \frac{1}{54}(54\cdot 24 - 4\cdot 81) = 18.$$

5. Comparison of results with the literature

In this section, we examine some of the most well-known classification results in the literature on product-quotient surfaces and compare them with the results obtained using our code. Specifically, we mention and discuss only the cases where there are discrepancies.

Remark 5.1. We compared our results with respect to those of [7,8] (for $K^2 = 8$) and those listed in the tables of [4] (for $1 \le K^2 \le 8$). We noticed that there are two mistakes since the authors forgot the possibility of exchanging the factors which provides only one irreducible family of surfaces instead of two, so $N = 1$, in the cases $G = \mathbb{Z}_5^2$ and $G = \mathbb{Z}_5^2 \rtimes \mathbb{Z}_3$.

The mistake found for $G = \mathbb{Z}_5^2$ was already mentioned in [1, Rem. 3.2 (3)], while, to our knowledge, that for $G = \mathbb{Z}_5^2 \rtimes \mathbb{Z}_3$ has never been noticed.

REMARK 5.2. Comparing the results for $K^2 = 0$ with respect those of [5], we noticed that [5, Table 1] does not contain the following two other cases:

Sing(X)	t_1	t_2	Id(G)
$1/4, 1/2^4, 3/4$	$2, 4, 6$	$2, 4, 6$	$\langle 72, 40\rangle$
$1/4, 1/2^4, 3/4$	$2, 4, 5$	$2, 4, 6$	$\langle 120, 34\rangle$

TABLE 6.

We verified that the MAGMA script of [5] returns also these results, so the authors just forgot to include them in their list.

We point out also that our code returns other three results than those of [5, Table 1] and Table 6, listed in Table 7.

Sing(X)	t_1	t_2	Id(G)
$2/5, 1/2^4, 3/5$	$2, 4, 5$	$2, 4, 5$	$\langle 160, 234\rangle$
$1/3^2, 1/2^2, 2/3^2$	$3, 3, 4$	$3, 3, 4$	$\langle 48, 3\rangle$
$1/3^2, 1/2^2, 2/3^2$	$3, 3, 4$	$2, 3, 7$	$\langle 168, 42\rangle$

TABLE 7.

These cases were not listed in [5, Table 1] since they do not provide surfaces of general type. Indeed, the invariant $\xi(X)$ is respectively equal to $\frac{1}{3}$, $\frac{2}{5}$, and $\frac{2}{5}$ for such cases, so that $\xi(X) < \frac{1}{2}$ and by [5, Thm. 5.3 and Cor. 5.4] they cannot give surfaces of general type.

We also excluded manually the secondary output of *ListGroups*(0, 1) (with a similar approach such as that explained in Section 3 for the case $(K^2, \chi) = (32, 4)$) to prove the following theorem.

THEOREM 5.3. *Let S be a product-quotient surface with $K_S^2 = p_g(S) = q(S) = 0$, then one of the following holds:*

(1) *S realizes one of the families of surfaces described in* [5, Table 1], *Table 6, and Table 7. Furthermore, all these surfaces are not of general type;*

(2) *S is the surface described in* [5, Prop. 7.1]. *In particular, it is a surface of general type whose minimal model is a numerical Godeaux surface with torsion of order* 4.

REMARK 5.4. Regarding the classification obtained for $K^2 = -1$, we get one case more than those two found in [5], see Table 8. This happened because the script developed in [5] looks for only surfaces of general type and so automatically excludes cases with $\xi(X) < \frac{1}{2}$. However, the last case found by us has $\xi(X) = \frac{2}{5}$ and so has been automatically excluded.

Sing(X)	t_1	t_2	Id(G)
$1/5, 2/5^2, 4/5$	$2, 5, 5$	$3, 3, 5$	$\langle 60, 5\rangle$
$1/5, 1/2^4, 4/5$	$2, 4, 5$	$2, 4, 5$	$\langle 160, 234\rangle$
$1/5^5$	$5, 5, 5$	$5, 5, 5$	$\langle 25, 2\rangle$

TABLE 8.

Furthermore, we found two irreducible families sharing the same algebraic data of the group $\mathbb{Z}_5^2$ instead of only one family found in [5].

We have also excluded manually the secondary output of *ListGroups*$(-1, 1)$ to prove the following theorem.

THEOREM 5.5. *Let S be a product-quotient surface with $K_S^2 = -1$, $p_g(S) = q(S) = 0$. Then, S realizes one of the families of surfaces described in Table 8. Furthermore, the first two cases of the table give product-quotient surfaces that are not of general type. Instead, the last case with group $\mathbb{Z}_5^2$ gives two irreducible families of surfaces that are not minimal and whose minimal model is a numerical Godeaux surface with torsion of order 5.*

APPENDIX

In this appendix, we list all regular product-quotient surfaces S of general type with $23 \leq K_S^2 \leq 32$ and $p_g(S) = 3$. In particular, we list the following information in the columns of Tables 9 to 21:

- K_S^2 is the self-intersection of the canonical class of S.
- G is the group, and Id is the identifier of the group in the MAGMA database of small groups; hence, the pair $\langle d, n\rangle$ of each row denotes that G is the n-th group of order d in the MAGMA database of small groups. Whenever G does not have an easy description, we simply denote it by $G(d, n)$, the group in the MAGMA database having identifier $\langle d, n\rangle$.
- Sing(X) is the singular locus of the quotient model $X := (C_1 \times C_2)/G$ defining the product-quotient surface S. It is given as a sequence of rational numbers with multiplicities, describing the types of cyclic quotient singularities. For instance, $3/5^4$ means 4 singular points of type $\frac{1}{5}(1, 3)$.
- t_1 and t_2 are the signatures of the corresponding spherical systems of generators, cf. Definition 1.3.
- N is the number of irreducible families. Indeed our tables have 555 lines, but we collect in the same line N families, which share all the other data. We employ the symbol ? whenever we are unable to determine the exact number of families in a row due to computational time constraints or machine memory overflow.
- deg(Φ_S) contains, for each family of the row, the degree of the canonical map of a surface S belonging to that family, whenever the computation of the degree can be

done. For example, if there are N irreducible families in a row, where $N=3$, and the degrees listed in the $\deg(\Phi_S)$ box for that row are 12 and 16, it indicates that the degree of the canonical map has been computed for surfaces from only two of the three families. Specifically, the degree is 12 for one family and 16 for the other. Furthermore, since the degree of the canonical map is not a topological invariant, then it may happen that surfaces belonging to the same family have distinct degrees of the canonical map. In this case, we simply list sequentially all degrees of the canonical map of the surfaces belonging to that family. For instance, suppose $\deg(\Phi_S)$ of a row is $12, (18, 16), 18$. This means the surfaces of two of these three families have a degree of the canonical map that is constant on the family and equal respectively to 12 and 18, while the other family has surfaces with a degree of the canonical map equal to either 18 or 16.

The number 0 means that the image of Φ_S has dimension 1.

For the groups occurring in Tables 9 to 21, we use the following notation:

$\mathbb{Z}_n^k$ is k-times the cyclic group of order n.

S_n is the symmetric group of n letters.

$\mathcal{A}_n$ is the alternating group.

D_n is the dihedral group of symmetries of the n-gon.

$\mathrm{ASL}(n,k)$ is the affine special linear group of $\mathbb{Z}_k^n$.

$\mathrm{PSL}(2,n)$ is the group of 2×2 matrices over $\mathbb{Z}_n$ with determinant 1 modulo the subgroup generated by $-\mathrm{Id}$.

$\mathrm{SO}(3,7)$ is the group of 3×3 orthogonal matrices over $\mathbb{F}_7$ with determinant 1.

He3 is the Heisenberg group of order 27:

$$\mathrm{He3} := \langle x, y, z | z^{-1}xyx^{-1}y^{-1}, x^3, y^3, z^3, xz = zx, yz = zy\rangle.$$

A 3-dimensional representation of He3 (over the field $\mathbb{Z}_3$) is given by sending

$$x \mapsto \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Q_8 is the quaternion group:

$$Q_8 := \langle x, y | x^4, x^2y^{-2}, y^{-1}xyx\rangle.$$

$K \wr H$ is the wreath product, so it is the semidirect product $K^H \rtimes H$, where K^H is the set of *functions* $f\colon H \to K$, with a group operation given by pointwise multiplication. Here, H is acting on K^H via left multiplication:

$$h \cdot f := f \circ h^{-1}, \quad f\colon H \to K \in K^H.$$

No.	K_S^2	Sing(X)	t_1	t_2	G	Id	N	deg(Φ_S)
1	32		2^6	2^8	$\mathbb{Z}_2^3$	$\langle 8, 5\rangle$	3	$8, 16^2$
2	32		2^5	2^{12}	$\mathbb{Z}_2^3$	$\langle 8, 5\rangle$	3	$0, 4, 8$
3	32		3^4	3^7	$\mathbb{Z}_3^2$	$\langle 9, 2\rangle$	2	$6, 12$
4	32		3^5	3^5	$\mathbb{Z}_3^2$	$\langle 9, 2\rangle$	1	9
5	32		$2^3, 4^2$	$2^3, 4^2$	$\mathbb{Z}_2^2 \rtimes \mathbb{Z}_4$	$\langle 16, 3\rangle$	2	16
6	32		$2^2, 4^2$	$2^2, 4^4$	$\mathbb{Z}_2^2 \rtimes \mathbb{Z}_4$	$\langle 16, 3\rangle$	2	
7	32		$2^2, 4^2$	$2^5, 4^2$	$\mathbb{Z}_2^2 \rtimes \mathbb{Z}_4$	$\langle 16, 3\rangle$	6	8
8	32		2^5	$2^5, 4^2$	$\mathbb{Z}_2 \times D_4$	$\langle 16, 11\rangle$	4	
9	32		$2^3, 4$	2^{12}	$\mathbb{Z}_2 \times D_4$	$\langle 16, 11\rangle$	6	0
10	32		$2^3, 4^2$	2^6	$\mathbb{Z}_2 \times D_4$	$\langle 16, 11\rangle$	2	
11	32		$2^2, 4^4$	2^5	$\mathbb{Z}_2 \times D_4$	$\langle 16, 11\rangle$	1	
12	32		2^6	2^6	$\mathbb{Z}_2 \times D_4$	$\langle 16, 11\rangle$	1	32
13	32		2^5	2^8	$\mathbb{Z}_2^4$	$\langle 16, 14\rangle$	13	$0, 8^5, 16^7$
14	32		2^6	2^6	$\mathbb{Z}_2^4$	$\langle 16, 14\rangle$	6	$8, 16^3, 32^2$
15	32		2^{12}	$3, 4^2$	S_4	$\langle 24, 12\rangle$	?	
16	32		$2^4, 3$	4^4	S_4	$\langle 24, 12\rangle$	1	
17	32		$2, 3, 4^2$	2^6	S_4	$\langle 24, 12\rangle$	1	
18	32		$2^2, 3^2$	$2^2, 4^4$	S_4	$\langle 24, 12\rangle$	1	
19	32		2^5	$2^5, 6$	$\mathbb{Z}_2^2 \times S_3$	$\langle 24, 14\rangle$	1	
20	32		$2^2, 4^2$	4^4	$G(32, 6)$	$\langle 32, 6\rangle$	1	
21	32		$2^2, 4^2$	$2^3, 4^2$	$G(32, 22)$	$\langle 32, 22\rangle$	7	16
22	32		$2^2, 4^4$	$2^3, 4$	$\mathbb{Z}_2^2 \wr \mathbb{Z}_2$	$\langle 32, 27\rangle$	2	
23	32		$2^3, 4$	$2^5, 4^2$	$\mathbb{Z}_2^2 \wr \mathbb{Z}_2$	$\langle 32, 27\rangle$	30	
24	32		$2^2, 4^2$	$2^3, 4^2$	$\mathbb{Z}_2^2 \wr \mathbb{Z}_2$	$\langle 32, 27\rangle$	1	
25	32		$2^3, 4^2$	2^5	$\mathbb{Z}_2^2 \wr \mathbb{Z}_2$	$\langle 32, 27\rangle$	4	
26	32		$2^2, 4^2$	2^6	$\mathbb{Z}_2^2 \wr \mathbb{Z}_2$	$\langle 32, 27\rangle$	4	
27	32		$2^2, 4^4$	$2^3, 4$	$\mathbb{Z}_2^2 \rtimes D_4$	$\langle 32, 28\rangle$	1	
28	32		2^5	2^6	$\mathbb{Z}_2^2 \times D_4$	$\langle 32, 46\rangle$	4	24
29	32		$2^3, 4^2$	2^5	$\mathbb{Z}_2^2 \times D_4$	$\langle 32, 46\rangle$	2	
30	32		$2^3, 4^2$	2^5	$Q_8 \rtimes \mathbb{Z}_2^2$	$\langle 32, 49\rangle$	1	
31	32		$2^2, 4, 12$	$2^2, 4^2$	$D_6 \rtimes \mathbb{Z}_4$	$\langle 48, 14\rangle$	1	
32	32		$2^2, 4^4$	$3, 4^2$	$\mathcal{A}_4 \rtimes \mathbb{Z}_4$	$\langle 48, 30\rangle$	3	
33	32		$2^3, 4$	$2^5, 6$	$S_3 \times D_4$	$\langle 48, 38\rangle$	1	
34	32		$4^2, 6$	2^6	$\mathbb{Z}_2 \times S_4$	$\langle 48, 48\rangle$	3	
35	32		$2, 4, 6$	2^{12}	$\mathbb{Z}_2 \times S_4$	$\langle 48, 48\rangle$	?	
36	32		$2^2, 4^2$	$2^4, 3$	$\mathbb{Z}_2 \times S_4$	$\langle 48, 48\rangle$	2	
37	32		$2^2, 4^2$	$2^2, 6^2$	$\mathbb{Z}_2 \times S_4$	$\langle 48, 48\rangle$	1	
38	32		$2^2, 4^4$	$2^3, 3$	$\mathbb{Z}_2 \times S_4$	$\langle 48, 48\rangle$	4	
39	32		$2^3, 6$	4^4	$\mathbb{Z}_2 \times S_4$	$\langle 48, 48\rangle$	1	
40	32		$2^3, 4^2$	$2^3, 6$	$\mathbb{Z}_2 \times S_4$	$\langle 48, 48\rangle$	1	
41	32		$2, 3, 4^2$	2^5	$\mathbb{Z}_2 \times S_4$	$\langle 48, 48\rangle$	1	
42	32		7^3	7^3	$\mathbb{Z}_7^2$	$\langle 49, 2\rangle$	7	$0, 5, 7, 10, 11, 14^2$

TABLE 9. Minimal product-quotient surfaces of general type with $q = 0$, $p_g = 3$, and $K^2 = 32$.

No.	K_S^2	Sing(X)	t_1	t_2	G	Id	N	deg(Φ_S)
43	32		$2, 5^2$	3^7	$\mathcal{A}_5$	$\langle 60, 5\rangle$	2	
44	32		2^8	$3^2, 5$	$\mathcal{A}_5$	$\langle 60, 5\rangle$	1	
45	32		$2^4, 3$	5^3	$\mathcal{A}_5$	$\langle 60, 5\rangle$	1	
46	32		3^4	5^3	$\mathcal{A}_5$	$\langle 60, 5\rangle$	1	
47	32		2^6	$3, 5^2$	$\mathcal{A}_5$	$\langle 60, 5\rangle$	2	
48	32		$2^2, 4^2$	$2^2, 4^2$	$G(64, 60)$	$\langle 64, 60\rangle$	3	32
49	32		$2^2, 4^2$	$2^2, 4^2$	$\mathbb{Z}_4 \rtimes (\mathbb{Z}_2^2 \rtimes \mathbb{Z}_4)$	$\langle 64, 71\rangle$	1	
50	32		$2^3, 4$	2^6	$G(64, 73)$	$\langle 64, 73\rangle$	1	
51	32		$2^3, 4$	$2^3, 4^2$	$G(64, 73)$	$\langle 64, 73\rangle$	4	
52	32		$2^2, 4^2$	$2^2, 4^2$	$G(64, 75)$	$\langle 64, 75\rangle$	1	
53	32		$2^3, 4$	4^4	$\mathbb{Z}_2 \wr \mathbb{Z}_2^2$	$\langle 64, 138\rangle$	1	
54	32		$2^3, 4$	$2^3, 4^2$	$\mathbb{Z}_2 \wr \mathbb{Z}_2^2$	$\langle 64, 138\rangle$	6	
55	32		2^5	2^5	$G(64, 211)$	$\langle 64, 211\rangle$	1	
56	32		2^5	2^5	$\mathbb{Z}_2^2 \times D_8$	$\langle 64, 250\rangle$	1	
57	32		$2^2, 4, 12$	$2^3, 4$	$\mathbb{Z}_2^2 \rtimes D_{12}$	$\langle 96, 89\rangle$	1	
58	32		$2^2, 4^2$	$4^2, 6$	$\mathrm{GL}(2, \mathbb{Z}_4)$	$\langle 96, 195\rangle$	1	
59	32		$2, 4, 6$	$2^2, 4^4$	$\mathrm{GL}(2, \mathbb{Z}_4)$	$\langle 96, 195\rangle$	10	
60	32		$2^2, 4^2$	$2^3, 6$	$\mathbb{Z}_2^2 \times S_4$	$\langle 96, 226\rangle$	1	
61	32		$2^3, 4^2$	$3, 4^2$	$\mathbb{Z}_2^2 \rtimes S_4$	$\langle 96, 227\rangle$	1	
62	32		$2^3, 3$	4^4	$\mathbb{Z}_2^2 \rtimes S_4$	$\langle 96, 227\rangle$	3	
63	32		2^6	$3, 4^2$	$\mathbb{Z}_2^2 \rtimes S_4$	$\langle 96, 227\rangle$	3	
64	32		$2^4, 5$	$3, 4^2$	S_5	$\langle 120, 34\rangle$	1	
65	32		$2, 5, 6$	4^4	S_5	$\langle 120, 34\rangle$	2	
66	32		$2, 5, 6$	$2^3, 4^2$	S_5	$\langle 120, 34\rangle$	1	
67	32		$2, 4, 5$	$2^2, 4^4$	$\mathbb{Z}_2^4 \rtimes D_5$	$\langle 160, 234\rangle$	?	
68	32		$3, 7^2$	4^3	$\mathrm{PSL}(2, 7)$	$\langle 168, 42\rangle$	4	
69	32		$3, 4^2$	7^3	$\mathrm{PSL}(2, 7)$	$\langle 168, 42\rangle$	1	
70	32		$2^2, 4^2$	$3^2, 7$	$\mathrm{PSL}(2, 7)$	$\langle 168, 42\rangle$	1	
71	32		$2^3, 4$	$4^2, 6$	$G(192, 955)$	$\langle 192, 955\rangle$	1	
72	32		$2, 4, 6$	$2^3, 4^2$	$G(192, 955)$	$\langle 192, 955\rangle$	7	
73	32		$2, 4, 6$	4^4	$G(192, 955)$	$\langle 192, 955\rangle$	2	
74	32		$2, 6^2$	$4^2, 10$	$\mathbb{Z}_2 \times S_5$	$\langle 240, 189\rangle$	1	
75	32		$2, 4, 6$	$2^2, 10^2$	$\mathbb{Z}_2 \times S_5$	$\langle 240, 189\rangle$	1	
76	32		4^3	4^3	$G(256, 295)$	$\langle 256, 295\rangle$	3	
77	32		4^3	4^3	$G(256, 298)$	$\langle 256, 298\rangle$	2	
78	32		4^3	4^3	$G(256, 306)$	$\langle 256, 306\rangle$	2	
79	32		$2, 6, 7$	$2, 8^2$	$\mathrm{SO}(3, 7)$	$\langle 336, 208\rangle$	2	
80	32		$2, 3, 14$	$2^2, 4^2$	$\mathbb{Z}_2 \times \mathrm{PSL}(2, 7)$	$\langle 336, 209\rangle$	1	
81	32		$2, 7, 14$	$3, 4^2$	$\mathbb{Z}_2 \times \mathrm{PSL}(2, 7)$	$\langle 336, 209\rangle$	1	
82	32		$2, 6, 7$	4^3	$\mathbb{Z}_2 \times \mathrm{PSL}(2, 7)$	$\langle 336, 209\rangle$	2	
83	32		$2, 6, 15$	$3, 4^2$	$\mathbb{Z}_3 \rtimes S_5$	$\langle 360, 120\rangle$	1	
84	32		$2, 4, 6$	$4^2, 10$	$\mathbb{Z}_2^2 \rtimes S_5$	$\langle 480, 951\rangle$	2	
85	32		$2, 3, 9$	7^3	$\mathrm{PSL}(2, 8)$	$\langle 504, 156\rangle$	6	
86	32		$2, 5^2$	$3^2, 11$	$\mathrm{PSL}(2, 11)$	$\langle 660, 13\rangle$	2	

TABLE 10. Minimal product-quotient surfaces of general type with $q = 0$, $p_g = 3$, and $K^2 = 32$.

No.	K_S^2	Sing(X)	t_1	t_2	G	Id	N	deg(Φ_S)
87	30	$1/2^2$	$2^3, 4$	$2^{10}, 4$	$\mathbb{Z}_2 \times D_4$	$\langle 16, 11\rangle$	6	0
88	30	$1/2^2$	$2^4, 4$	$2^5, 4$	$\mathbb{Z}_2 \times D_4$	$\langle 16, 11\rangle$	2	4
89	30	$1/2^2$	$2^3, 8$	$2^5, 4$	$\mathbb{Z}_2 \times D_8$	$\langle 32, 39\rangle$	1	
90	30	$1/2^2$	$2^3, 12$	$2^4, 4$	$S_3 \times D_4$	$\langle 48, 38\rangle$	1	
91	30	$1/2^2$	$2^3, 4$	$2^3, 6, 12$	$S_3 \times D_4$	$\langle 48, 38\rangle$	1	
92	30	$1/2^2$	$2, 4, 6$	$2^{10}, 4$	$\mathbb{Z}_2 \times S_4$	$\langle 48, 48\rangle$	?	
93	30	$1/2^2$	$2^2, 3, 4$	$2^4, 4$	$\mathbb{Z}_2 \times S_4$	$\langle 48, 48\rangle$	2	
94	30	$1/2^2$	$2, 3^6$	$2, 5^2$	$\mathcal{A}_5$	$\langle 60, 5\rangle$	1	
95	30	$1/2^2$	$2^2, 4, 8$	$2^3, 8$	$(\mathbb{Z}_2 \times D_8) \rtimes \mathbb{Z}_2$	$\langle 64, 128\rangle$	2	
96	30	$1/2^2$	$2, 6, 12$	$2^2, 3, 4$	$S_3 \times S_4$	$\langle 144, 183\rangle$	1	
97	30	$1/2^2$	$2, 7^3$	$3^2, 4$	PSL(2, 7)	$\langle 168, 42\rangle$	4	
98	30	$1/2^2$	$3^2, 4$	$3^3, 6$	ASL(2, 3)	$\langle 216, 153\rangle$	4	
99	30	$1/2^2$	$2, 4, 10$	$2^2, 3, 4$	$\mathbb{Z}_2 \times S_5$	$\langle 240, 189\rangle$	1	
100	30	$1/2^2$	$2, 9^2$	$3^2, 6$	$G(324, 160)$	$\langle 324, 160\rangle$	3	
101	30	$1/2^2$	$2, 4, 7$	$4, 6^2$	$\mathbb{Z}_2 \times$ PSL(2, 7)	$\langle 336, 209\rangle$	2	
102	30	$1/2^2$	$2, 4, 5$	$4, 6^2$	$\mathbb{Z}_2 \times \mathcal{A}_6$	$\langle 720, 766\rangle$	2	
103	29	$1/3, 2/3$	$2^{10}, 3$	$3, 4^2$	S_4	$\langle 24, 12\rangle$	?	
104	29	$1/3, 2/3$	$2^3, 4^2, 6$	$3, 4^2$	$\mathcal{A}_4 \rtimes \mathbb{Z}_4$	$\langle 48, 30\rangle$	3	
105	29	$1/3, 2/3$	$3, 4^2$	$4^4, 6$	$\mathcal{A}_4 \rtimes \mathbb{Z}_4$	$\langle 48, 30\rangle$	1	
106	29	$1/3, 2/3$	$2, 4, 6$	$2^{10}, 3$	$\mathbb{Z}_2 \times S_4$	$\langle 48, 48\rangle$	?	
107	29	$1/3, 2/3$	$2^3, 3$	$4^4, 6$	$\mathbb{Z}_2 \times S_4$	$\langle 48, 48\rangle$	2	
108	29	$1/3, 2/3$	$2^3, 3$	$2^3, 4^2, 6$	$\mathbb{Z}_2 \times S_4$	$\langle 48, 48\rangle$	4	
109	29	$1/3, 2/3$	$2, 4, 6$	$4^4, 6$	GL$(2, \mathbb{Z}_4)$	$\langle 96, 195\rangle$	1	
110	29	$1/3, 2/3$	$2, 4, 6$	$2^3, 4^2, 6$	GL$(2, \mathbb{Z}_4)$	$\langle 96, 195\rangle$	8	
111	29	$1/3, 2/3$	$2^3, 3, 4$	$3, 4^2$	$G(96, 227)$	$\langle 96, 227\rangle$	3	
112	29	$1/3, 2/3$	$2, 3, 8$	$4^4, 6$	$G(192, 181)$	$\langle 192, 181\rangle$	1	
113	29	$1/3, 2/3$	$2, 4, 6$	$3, 4^3$	$G(192, 955)$	$\langle 192, 955\rangle$	2	
114	29	$1/3, 2/3$	$2^3, 3$	$4, 6, 8$	$G(192, 956)$	$\langle 192, 956\rangle$	1	
115	29	$1/3, 2/3$	$2^3, 3$	$4, 6, 8$	$G(192, 1494)$	$\langle 192, 1494\rangle$	1	
116	29	$1/3, 2/3$	$2, 4, 6$	$2^2, 6, 10$	$\mathbb{Z}_2 \times S_5$	$\langle 240, 189\rangle$	2	
117	29	$1/3, 2/3$	$2, 4, 6$	$4, 6, 8$	$G(384, 5602)$	$\langle 384, 5602\rangle$	2	
118	29	$1/3, 2/3$	$2, 3, 10$	$2, 4, 12$	$G(1320, 133)$	$\langle 1320, 133\rangle$	4	
119	28	$1/2^4$	$2^2, 4^2$	$2^8, 4^2$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\langle 8, 2\rangle$	1	0
120	28	$1/2^4$	2^5	2^{11}	$\mathbb{Z}_2^3$	$\langle 8, 5\rangle$	6	$0^2, 4^3, 8$
121	28	$1/2^4$	$2^3, 4^3$	2^5	$\mathbb{Z}_2 \times D_4$	$\langle 16, 11\rangle$	3	
122	28	$1/2^4$	$2^3, 4$	$2^8, 4^2$	$\mathbb{Z}_2 \times D_4$	$\langle 16, 11\rangle$	5	
123	28	$1/2^4$	$2^3, 4$	2^{11}	$\mathbb{Z}_2 \times D_4$	$\langle 16, 11\rangle$	14	0
124	28	$1/2^4$	2^5	$2^6, 4$	$\mathbb{Z}_2 \times D_4$	$\langle 16, 11\rangle$	6	8
125	28	$1/2^4$	$2^2, 3^2$	$3^4, 6^2$	$\mathbb{Z}_3 \times S_3$	$\langle 18, 3\rangle$	6	6^2
126	28	$1/2^4$	$2^2, 3^5$	$3, 6^2$	$\mathbb{Z}_3 \times S_3$	$\langle 18, 3\rangle$	1	
127	28	$1/2^4$	$2^2, 3^2$	$2^2, 3^5$	$\mathbb{Z}_3 \rtimes S_3$	$\langle 18, 4\rangle$	2	
128	28	$1/2^4$	$2^2, 3^2$	$2^3, 4^3$	S_4	$\langle 24, 12\rangle$	1	

TABLE 11. Minimal product-quotient surfaces of general type with $q = 0$, $p_g = 3$, and $K^2 \in \{30, 29, 28\}$.

No.	K_S^2	Sing(X)	t_1	t_2	G	Id	N	deg(Φ_S)
129	28	$1/2^4$	2^{11}	$3, 4^2$	S_4	$\langle 24, 12\rangle$	1	
130	28	$1/2^4$	$2^3, 6^2$	2^5	$\mathbb{Z}_2^2 \times S_3$	$\langle 24, 14\rangle$	3	
131	28	$1/2^4$	2^5	$2^5, 3$	$\mathbb{Z}_2^2 \times S_3$	$\langle 24, 14\rangle$	1	
132	28	$1/2^4$	$2, 4^2, 8$	$2^2, 4^2$	$\mathbb{Z}_4 \wr \mathbb{Z}_2$	$\langle 32, 11\rangle$	1	
133	28	$1/2^4$	$2^3, 4$	$2^3, 4^3$	$\mathbb{Z}_2^2 \wr \mathbb{Z}_2$	$\langle 32, 27\rangle$	4	
134	28	$1/2^4$	$2^3, 4$	$2^6, 4$	$\mathbb{Z}_2^2 \wr \mathbb{Z}_2$	$\langle 32, 27\rangle$	30	
135	28	$1/2^4$	$2^3, 4$	$2^3, 4^3$	$\mathbb{Z}_2^2 \rtimes D_4$	$\langle 32, 28\rangle$	4	
136	28	$1/2^4$	$2^4, 8$	2^5	$\mathbb{Z}_2 \times D_8$	$\langle 32, 39\rangle$	2	
137	28	$1/2^4$	$2, 4^2, 8$	2^5	$\mathbb{Z}_8 \rtimes \mathbb{Z}_2^2$	$\langle 32, 43\rangle$	1	
138	28	$1/2^4$	$2^2, 4, 6$	2^5	$\mathbb{Z}_2 \times D_{12}$	$\langle 48, 36\rangle$	1	
139	28	$1/2^4$	$2^3, 4$	$2^5, 3$	$S_3 \times D_4$	$\langle 48, 38\rangle$	1	
140	28	$1/2^4$	$2^2, 4, 6$	2^5	$S_3 \times D_4$	$\langle 48, 38\rangle$	2	
141	28	$1/2^4$	$2^3, 4$	$2^3, 6^2$	$S_3 \times D_4$	$\langle 48, 38\rangle$	2	
142	28	$1/2^4$	$2^2, 3, 4^2$	$2^3, 4$	$\mathbb{Z}_2 \times S_4$	$\langle 48, 48\rangle$	3	
143	28	$1/2^4$	$2^3, 3$	$2^3, 4^3$	$\mathbb{Z}_2 \times S_4$	$\langle 48, 48\rangle$	5	
144	28	$1/2^4$	$2^3, 4$	$4^2, 6^2$	$\mathbb{Z}_2 \times S_4$	$\langle 48, 48\rangle$	2	
145	28	$1/2^4$	$2, 4, 6$	2^{11}	$\mathbb{Z}_2 \times S_4$	$\langle 48, 48\rangle$	?	
146	28	$1/2^4$	$2, 4, 6$	$2^8, 4^2$	$\mathbb{Z}_2 \times S_4$	$\langle 48, 48\rangle$	?	
147	28	$1/2^4$	$2^2, 4, 6$	2^5	$\mathbb{Z}_2 \times S_4$	$\langle 48, 48\rangle$	2	
148	28	$1/2^4$	$2, 5^2$	$2^2, 3^5$	$\mathcal{A}_5$	$\langle 60, 5\rangle$	1	
149	28	$1/2^4$	$2^3, 4$	$2^4, 8$	$(\mathbb{Z}_2 \times D_8) \rtimes \mathbb{Z}_2$	$\langle 64, 128\rangle$	5	
150	28	$1/2^4$	$2, 4^2, 8$	$2^3, 4$	$D_4 \rtimes D_4$	$\langle 64, 134\rangle$	1	
151	28	$1/2^4$	$2, 4^2, 8$	$2^3, 4$	$(\mathbb{Z}_4 \wr \mathbb{Z}_2) \rtimes \mathbb{Z}_2$	$\langle 64, 135\rangle$	1	
152	28	$1/2^4$	$2^3, 16$	2^5	$\mathbb{Z}_2 \times D_{16}$	$\langle 64, 186\rangle$	1	
153	28	$1/2^4$	$2, 3^2, 4$	$2^2, 3^2$	$\mathbb{Z}_3 \rtimes S_4$	$\langle 72, 43\rangle$	1	
154	28	$1/2^4$	$2^2, 4, 6$	$2^3, 4$	$\mathbb{Z}_2^2 \rtimes D_{12}$	$\langle 96, 89\rangle$	1	
155	28	$1/2^4$	$2, 8, 12$	2^5	$(\mathrm{SL}(2,3) \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}_2$	$\langle 96, 193\rangle$	1	
156	28	$1/2^4$	$2, 4, 6$	$2^3, 4^3$	$\mathrm{GL}(2, \mathbb{Z}_4)$	$\langle 96, 195\rangle$	9	
157	28	$1/2^4$	$2^2, 4, 6$	$2^3, 4$	$\mathbb{Z}_2^2 \times S_4$	$\langle 96, 226\rangle$	2	
158	28	$1/2^4$	$2, 4, 5$	$3^4, 6^2$	S_5	$\langle 120, 34\rangle$	2	
159	28	$1/2^4$	$2^3, 4$	$5, 6^2$	S_5	$\langle 120, 34\rangle$	1	
160	28	$1/2^4$	$2, 4, 6$	$2^2, 5^3$	S_5	$\langle 120, 34\rangle$	1	
161	28	$1/2^4$	$2^2, 5, 10$	$2^3, 3$	$\mathbb{Z}_2 \times \mathcal{A}_5$	$\langle 120, 35\rangle$	1	
162	28	$1/2^4$	$2, 6, 10$	2^5	$\mathbb{Z}_2 \times \mathcal{A}_5$	$\langle 120, 35\rangle$	1	
163	28	$1/2^4$	$2^3, 4$	$2^3, 16$	$G(128, 916)$	$\langle 128, 916\rangle$	1	
164	28	$1/2^4$	$2, 4, 18$	2^5	$G(144, 109)$	$\langle 144, 109\rangle$	1	
165	28	$1/2^4$	$2^2, 4, 6$	$2^3, 3$	$S_3 \times S_4$	$\langle 144, 183\rangle$	1	
166	28	$1/2^4$	$2^2, 3, 12$	$2^3, 3$	$S_3 \times S_4$	$\langle 144, 183\rangle$	1	
167	28	$1/2^4$	$2, 4, 5$	$2^3, 4^3$	$\mathbb{Z}_2^4 \rtimes D_5$	$\langle 160, 234\rangle$	?	
168	28	$1/2^4$	$2, 3, 8$	4^5	$G(192, 181)$	$\langle 192, 181\rangle$	1	
169	28	$1/2^4$	$2, 5, 8$	$3, 6^2$	$\mathrm{SL}(2,5) \rtimes \mathbb{Z}_2$	$\langle 240, 90\rangle$	1	
170	28	$1/2^4$	$2, 4, 6$	$2^2, 5, 10$	$\mathbb{Z}_2 \times S_5$	$\langle 240, 189\rangle$	1	

TABLE 12. Minimal product-quotient surfaces of general type with $q = 0$, $p_g = 3$, and $K^2 = 28$.

No.	K_S^2	Sing(X)	t_1	t_2	G	Id	N	deg(Φ_S)
171	28	$1/2^4$	$2,6,10$	$2^3,4$	$\mathbb{Z}_2\times S_5$	$\langle 240,189\rangle$	2	
172	28	$1/2^4$	$2,4,8$	$3,6^2$	SO(3, 7)	$\langle 336,208\rangle$	2	
173	28	$1/2^4$	$2,4,8$	$2,6^2$	$\mathbb{Z}_2\times$ SO(3, 7)	$\langle 672,1254\rangle$	2	
174	28	$1/2^4$	$2,4,6$	$2,8^2$	$\mathbb{Z}_2\times$ SO(3, 7)	$\langle 672,1254\rangle$	2	
175	28	$3/5^2$	$2^3,5$	$3^3,5$	$\mathcal{A}_5$	$\langle 60,5\rangle$	2	
176	28	$3/5^2$	$2^6,5$	$3^2,5$	$\mathcal{A}_5$	$\langle 60,5\rangle$	1	
177	28	$3/5^2$	$2^3,5$	$3,6,10$	$\mathbb{Z}_2\times\mathcal{A}_5$	$\langle 120,35\rangle$	1	
178	28	$3/5^2$	$2,4,5$	$4^4,5$	$\mathbb{Z}_2^4\rtimes D_5$	$\langle 160,234\rangle$	?	
179	28	$3/5^2$	$4^2,5$	$4^2,5$	$\mathbb{Z}_2^4\rtimes D_5$	$\langle 160,234\rangle$	3	
180	28	$3/5^2$	$2,4,5$	$2^3,4^2,5$	$\mathbb{Z}_2^4\rtimes D_5$	$\langle 160,234\rangle$	?	
181	28	$3/5^2$	$2^3,5$	$4^2,5$	$\mathbb{Z}_2^4\rtimes D_5$	$\langle 160,234\rangle$	3	
182	28	$3/5^2$	$2^3,5$	$3^2,5$	$\mathcal{A}_6$	$\langle 360,118\rangle$	1	
183	28	$3/5^2$	$3^2,5$	$4^2,5$	$\mathcal{A}_6$	$\langle 360,118\rangle$	2	
184	28	$3/5^2$	$2,4,5$	$3^3,5$	$\mathcal{A}_6$	$\langle 360,118\rangle$	6	
185	28	$3/5^2$	$2,4,5$	$3,6,10$	$\mathbb{Z}_2\times\mathcal{A}_6$	$\langle 720,766\rangle$	2	
186	27	$1/5,4/5$	$2^3,5$	$3^3,5$	$\mathcal{A}_5$	$\langle 60,5\rangle$	2	
187	27	$1/5,4/5$	$2^6,5$	$3^2,5$	$\mathcal{A}_5$	$\langle 60,5\rangle$	1	
188	27	$1/5,4/5$	$2^3,5$	$3,6,10$	$\mathbb{Z}_2\times\mathcal{A}_5$	$\langle 120,35\rangle$	1	
189	27	$1/5,4/5$	$2,4,5$	$4^4,5$	$\mathbb{Z}_2^4\rtimes D_5$	$\langle 160,234\rangle$	7	
190	27	$1/5,4/5$	$4^2,5$	$4^2,5$	$\mathbb{Z}_2^4\rtimes D_5$	$\langle 160,234\rangle$	2	
191	27	$1/5,4/5$	$2,4,5$	$2^3,4^2,5$	$\mathbb{Z}_2^4\rtimes D_5$	$\langle 160,234\rangle$	?	
192	27	$1/5,4/5$	$2^3,5$	$4^2,5$	$\mathbb{Z}_2^4\rtimes D_5$	$\langle 160,234\rangle$	3	
193	27	$1/5,4/5$	$2^3,5$	$3^2,5$	$\mathcal{A}_6$	$\langle 360,118\rangle$	1	
194	27	$1/5,4/5$	$3^2,5$	$4^2,5$	$\mathcal{A}_6$	$\langle 360,118\rangle$	2	
195	27	$1/5,4/5$	$2,4,5$	$3^3,5$	$\mathcal{A}_6$	$\langle 360,118\rangle$	6	
196	27	$1/5,4/5$	$2,4,5$	$3,6,10$	$\mathbb{Z}_2\times\mathcal{A}_6$	$\langle 720,766\rangle$	2	
197	27	$1/3,1/2^2,2/3$	$2,4,6$	$2^8,3,4$	$\mathbb{Z}_2\times S_4$	$\langle 48,48\rangle$	?	
198	26	$1/2^6$	$2^3,4$	$2^9,4$	$\mathbb{Z}_2\times D_4$	$\langle 16,11\rangle$	14	0
199	26	$1/2^6$	$2^3,4$	$2^6,4^3$	$\mathbb{Z}_2\times D_4$	$\langle 16,11\rangle$	2	
200	26	$1/2^6$	$2,6^2$	$2^3,3^4$	$S_3\times S_3$	$\langle 36,10\rangle$	1	
201	26	$1/2^6$	$2,3^3,6^2$	$2^3,3$	$S_3\times S_3$	$\langle 36,10\rangle$	2	
202	26	$1/2^6$	$2^3,4$	$2^3,4,6$	$S_3\times D_4$	$\langle 48,38\rangle$	1	
203	26	$1/2^6$	$2^3,3,12$	$2^3,4$	$S_3\times D_4$	$\langle 48,38\rangle$	1	
204	26	$1/2^6$	$2,4,6$	$2^6,4^3$	$\mathbb{Z}_2\times S_4$	$\langle 48,48\rangle$	?	
205	26	$1/2^6$	$2^2,3^2,4$	$2^3,4$	$\mathbb{Z}_2\times S_4$	$\langle 48,48\rangle$	2	
206	26	$1/2^6$	$2^3,4$	$2^3,4,6$	$\mathbb{Z}_2\times S_4$	$\langle 48,48\rangle$	3	
207	26	$1/2^6$	$2,4,6$	$2^9,4$	$\mathbb{Z}_2\times S_4$	$\langle 48,48\rangle$	?	
208	26	$1/2^6$	$2,5^2$	$2^3,3^4$	$\mathcal{A}_5$	$\langle 60,5\rangle$	1	
209	26	$1/2^6$	$2^3,4$	$2^3,28$	$D_4\times D_7$	$\langle 112,31\rangle$	1	
210	26	$1/2^6$	$2,3^3,6^2$	$2,4,5$	S_5	$\langle 120,34\rangle$	1	
211	26	$1/2^6$	$2,4,6$	$2^2,4,10$	$\mathbb{Z}_2\times S_5$	$\langle 240,189\rangle$	2	
212	26	$1/2^6$	$2,6^2$	$2,7,8$	SO(3, 7)	$\langle 336,208\rangle$	2	
213	26	$1/4,1/2^2,3/4$	$2^3,4,8$	$2^3,8$	$\mathbb{Z}_2\times D_8$	$\langle 32,39\rangle$	2	
214	26	$1/4,1/2^2,3/4$	$2,4,5$	$3^4,4,6$	S_5	$\langle 120,34\rangle$	2	

TABLE 13. Minimal product-quotient surfaces of general type with $q=0$, $p_g=3$, and $K^2\in\{28,27,26\}$.

No.	K_S^2	Sing(X)	t_1	t_2	G	Id	N	deg(Φ_S)
215	26	$1/4, 1/2^2, 3/4$	$2, 4, 7$	$3^3, 4$	PSL(2, 7)	$\langle 168, 42\rangle$	2	
216	26	$1/4, 1/2^2, 3/4$	$2, 4, 7^2$	$3^2, 4$	PSL(2, 7)	$\langle 168, 42\rangle$	4	
217	26	$1/4, 1/2^2, 3/4$	$3^2, 4$	$3^3, 4$	ASL(2, 3)	$\langle 216, 153\rangle$	4	
218	26	$1/4, 1/2^2, 3/4$	$2, 4, 5$	$3^3, 4$	$\mathcal{A}_6$	$\langle 360, 118\rangle$	8	
219	26	$1/3^2, 2/3^2$	$2^8, 3^2$	$3, 4^2$	S_4	$\langle 24, 12\rangle$	1	
220	26	$1/3^2, 2/3^2$	$2^2, 3^2$	$2^3, 3, 4^2$	S_4	$\langle 24, 12\rangle$	1	
221	26	$1/3^2, 2/3^2$	$2^2, 3^2$	$3, 4^4$	S_4	$\langle 24, 12\rangle$	2	
222	26	$1/3^2, 2/3^2$	$2^4, 3$	$3^2, 4^2$	S_4	$\langle 24, 12\rangle$	2	
223	26	$1/3^2, 2/3^2$	$2^2, 3^2$	$2^4, 6^2$	$\mathbb{Z}_2 \times \mathcal{A}_4$	$\langle 24, 13\rangle$	2	
224	26	$1/3^2, 2/3^2$	$2, 6^2$	$2^8, 3^2$	$\mathbb{Z}_2 \times \mathcal{A}_4$	$\langle 24, 13\rangle$	1	
225	26	$1/3^2, 2/3^2$	$3, 9^2$	$3^2, 9^2$	$\mathbb{Z}_3 \times \mathbb{Z}_9$	$\langle 27, 2\rangle$	6	$6^3, 7, 9, 10$
226	26	$1/3^2, 2/3^2$	$2^4, 3$	$3, 8^2$	GL(2, 3)	$\langle 48, 29\rangle$	1	
227	26	$1/3^2, 2/3^2$	$2^3, 3, 4^2$	$3, 4^2$	$\mathcal{A}_4 \rtimes \mathbb{Z}_4$	$\langle 48, 30\rangle$	3	
228	26	$1/3^2, 2/3^2$	$2, 4^2, 6^2$	$3, 4^2$	$\mathcal{A}_4 \rtimes \mathbb{Z}_4$	$\langle 48, 30\rangle$	2	
229	26	$1/3^2, 2/3^2$	$3, 4^2$	$3, 4^4$	$\mathcal{A}_4 \rtimes \mathbb{Z}_4$	$\langle 48, 30\rangle$	2	
230	26	$1/3^2, 2/3^2$	$2^3, 3$	$3, 4^4$	$\mathbb{Z}_2 \times S_4$	$\langle 48, 48\rangle$	2	
231	26	$1/3^2, 2/3^2$	$2^3, 3^2$	$4^2, 6$	$\mathbb{Z}_2 \times S_4$	$\langle 48, 48\rangle$	1	
232	26	$1/3^2, 2/3^2$	$2, 4^2, 6^2$	$2^3, 3$	$\mathbb{Z}_2 \times S_4$	$\langle 48, 48\rangle$	3	
233	26	$1/3^2, 2/3^2$	$2, 4, 6$	$2^8, 3^2$	$\mathbb{Z}_2 \times S_4$	$\langle 48, 48\rangle$	?	
234	26	$1/3^2, 2/3^2$	$2^2, 3, 4$	$2^4, 3$	$\mathbb{Z}_2 \times S_4$	$\langle 48, 48\rangle$	1	
235	26	$1/3^2, 2/3^2$	$2^3, 3$	$2^3, 3, 4^2$	$\mathbb{Z}_2 \times S_4$	$\langle 48, 48\rangle$	3	
236	26	$1/3^2, 2/3^2$	$2^2, 3, 4$	$2^2, 6^2$	$\mathbb{Z}_2 \times S_4$	$\langle 48, 48\rangle$	1	
237	26	$1/3^2, 2/3^2$	$2, 6^2$	$2^4, 6^2$	$\mathbb{Z}_2^2 \times \mathcal{A}_4$	$\langle 48, 49\rangle$	5	8
238	26	$1/3^2, 2/3^2$	$2^2, 3^2$	$2^3, 3^2$	$\mathbb{Z}_2^4 \rtimes \mathbb{Z}_3$	$\langle 48, 50\rangle$	2	
239	26	$1/3^2, 2/3^2$	$2^3, 3^2$	$3, 5^2$	$\mathcal{A}_5$	$\langle 60, 5\rangle$	2	
240	26	$1/3^2, 2/3^2$	$2^6, 3$	$3^2, 5$	$\mathcal{A}_5$	$\langle 60, 5\rangle$	1	
241	26	$1/3^2, 2/3^2$	$2^2, 3^2$	$3, 8^2$	$G(96, 64)$	$\langle 96, 64\rangle$	2	
242	26	$1/3^2, 2/3^2$	$2, 6^2$	$3^2, 4^2$	$(\mathbb{Z}_2^2 \wr \mathbb{Z}_2) \rtimes \mathbb{Z}_3$	$\langle 96, 70\rangle$	2	
243	26	$1/3^2, 2/3^2$	$2^2, 3^2$	$4, 6^2$	$G(96, 72)$	$\langle 96, 72\rangle$	2	
244	26	$1/3^2, 2/3^2$	$2, 6, 8$	$2^2, 6^2$	$\mathbb{Z}_2 \times$ GL(2, 3)	$\langle 96, 189\rangle$	1	
245	26	$1/3^2, 2/3^2$	$2^2, 3, 4$	$4^2, 6$	GL(2, $\mathbb{Z}_4$)	$\langle 96, 195\rangle$	2	
246	26	$1/3^2, 2/3^2$	$2, 4, 6$	$2, 4^2, 6^2$	GL(2, $\mathbb{Z}_4$)	$\langle 96, 195\rangle$	1	
247	26	$1/3^2, 2/3^2$	$2, 4, 6$	$3, 4^4$	GL(2, $\mathbb{Z}_4$)	$\langle 96, 195\rangle$	1	
248	26	$1/3^2, 2/3^2$	$2, 4, 6$	$2^3, 3, 4^2$	GL(2, $\mathbb{Z}_4$)	$\langle 96, 195\rangle$	8	
249	26	$1/3^2, 2/3^2$	$2^3, 3$	$3^2, 4^2$	$G(96, 227)$	$\langle 96, 227\rangle$	4	
250	26	$1/3^2, 2/3^2$	$3, 4^2$	$3^2, 4^2$	$\mathbb{Z}_2^2 \rtimes S_4$	$\langle 96, 227\rangle$	2	
251	26	$1/3^2, 2/3^2$	$2^3, 3^2$	$3, 4^2$	$\mathbb{Z}_2^2 \rtimes S_4$	$\langle 96, 227\rangle$	2	
252	26	$1/3^2, 2/3^2$	$2, 5, 6$	$3^2, 4^2$	S_5	$\langle 120, 34\rangle$	2	
253	26	$1/3^2, 2/3^2$	$2^2, 6^2$	$3, 4^2$	S_5	$\langle 120, 34\rangle$	1	
254	26	$1/3^2, 2/3^2$	$2^2, 3, 4$	$3^2, 7$	PSL(2, 7)	$\langle 168, 42\rangle$	1	
255	26	$1/3^2, 2/3^2$	$2, 3, 8$	$2, 4^2, 6^2$	$G(192, 181)$	$\langle 192, 181\rangle$	2	
256	26	$1/3^2, 2/3^2$	$2, 3, 8$	$3, 4^4$	$G(192, 181)$	$\langle 192, 181\rangle$	1	
257	26	$1/3^2, 2/3^2$	$2, 6^2$	$4, 6^2$	$\mathbb{Z}_2 \wr \mathcal{A}_4$	$\langle 192, 201\rangle$	3	
258	26	$1/3^2, 2/3^2$	$2^2, 3, 4$	$3, 4^2$	$\mathbb{Z}_2^3 \rtimes S_4$	$\langle 192, 1493\rangle$	3	

TABLE 14. Minimal product-quotient surfaces of general type with $q = 0$, $p_g = 3$, and $K^2 = 26$.

No.	K_S^2	Sing(X)	t_1	t_2	G	Id	N	deg(Φ_S)
259	26	$1/3^2, 2/3^2$	$2^3, 3$	$3, 8^2$	$G(192, 1494)$	$\langle 192, 1494\rangle$	1	
260	26	$1/3^2, 2/3^2$	$2, 5, 6$	$3, 8^2$	$\mathrm{SL}(2,5) \rtimes \mathbb{Z}_2$	$\langle 240, 90\rangle$	1	
261	26	$1/3^2, 2/3^2$	$2, 12^2$	$3, 4^2$	$\mathcal{A}_5 \rtimes \mathbb{Z}_4$	$\langle 240, 91\rangle$	1	
262	26	$1/3^2, 2/3^2$	$2, 6^2$	$4^2, 6$	$\mathbb{Z}_2 \times S_5$	$\langle 240, 189\rangle$	1	
263	26	$1/3^2, 2/3^2$	$3^2, 4$	$4, 6^2$	$G(384, 4)$	$\langle 384, 4\rangle$	2	
264	26	$1/3^2, 2/3^2$	$2, 3, 11$	$3, 5^2$	$\mathrm{PSL}(2, 11)$	$\langle 660, 13\rangle$	6	
265	26	$1/3^2, 2/3^2$	$2, 3, 7$	$3, 13^2$	$\mathrm{PSL}(2, 13)$	$\langle 1092, 25\rangle$	12	
266	26	$1/3^2, 2/3^2$	$2, 3, 7$	$3, 8^2$	$G(1344, 814)$	$\langle 1344, 814\rangle$	8	
267	26	$1/4, 1/2^2, 3/4$	$3^2, 4$	$3^2, 4$	$G(1944, 3875)$	$\langle 1944, 3875\rangle$	2	
268	25	$1/3, 1/2^4, 2/3$	$2^9, 3$	$3, 4^2$	S_4	$\langle 24, 12\rangle$	1	
269	25	$1/3, 1/2^4, 2/3$	$2, 4, 6$	$2^6, 3, 4^2$	$\mathbb{Z}_2 \times S_4$	$\langle 48, 48\rangle$	?	
270	25	$1/3, 1/2^4, 2/3$	$2^3, 3$	$2^4, 4, 6$	$\mathbb{Z}_2 \times S_4$	$\langle 48, 48\rangle$	4	
271	25	$1/3, 1/2^4, 2/3$	$2, 4, 6$	$2^9, 3$	$\mathbb{Z}_2 \times S_4$	$\langle 48, 48\rangle$	?	
272	25	$1/3, 1/2^4, 2/3$	$2, 4^3, 6$	$2^3, 3$	$\mathbb{Z}_2 \times S_4$	$\langle 48, 48\rangle$	2	
273	25	$1/3, 1/2^4, 2/3$	$2^2, 8, 12$	$2^3, 3$	$G(96, 193)$	$\langle 96, 193\rangle$	1	
274	25	$1/3, 1/2^4, 2/3$	$2, 4, 6$	$2^4, 4, 6$	$\mathrm{GL}(2, \mathbb{Z}_4)$	$\langle 96, 195\rangle$	6	
275	25	$1/3, 1/2^4, 2/3$	$2, 4, 6$	$2, 4^3, 6$	$\mathrm{GL}(2, \mathbb{Z}_4)$	$\langle 96, 195\rangle$	1	
276	25	$1/3, 1/2^4, 2/3$	$2, 4, 6$	$2^2, 3, 5^2$	S_5	$\langle 120, 34\rangle$	1	
277	25	$1/3, 1/2^4, 2/3$	$2^2, 5, 6$	$3, 4^2$	S_5	$\langle 120, 34\rangle$	1	
278	25	$1/3, 1/2^4, 2/3$	$2^2, 5, 6$	$2^3, 3$	$\mathbb{Z}_2 \times \mathcal{A}_5$	$\langle 120, 35\rangle$	1	
279	25	$1/3, 1/2^4, 2/3$	$2, 3, 8$	$2, 4^3, 6$	$G(192, 181)$	$\langle 192, 181\rangle$	3	
280	25	$1/3, 1/2^4, 2/3$	$2, 4, 6$	$2^2, 5, 6$	$\mathbb{Z}_2 \times S_5$	$\langle 240, 189\rangle$	2	
281	25	$1/3, 1/2^4, 2/3$	$2, 4, 6$	$2, 10, 12$	$\mathbb{Z}_2^2 \rtimes S_5$	$\langle 480, 951\rangle$	2	
282	25	$1/3, 2/5^2, 2/3$	$2, 6, 10$	$2^2, 3, 5$	$\mathbb{Z}_2 \times \mathcal{A}_5$	$\langle 120, 35\rangle$	1	
283	24	$1/2^8$	2^6	2^{10}	$\mathbb{Z}_2^2$	$\langle 4, 2\rangle$	1	0
284	24	$1/2^8$	$2^3, 4^2$	$2^4, 4^2$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\langle 8, 2\rangle$	1	8
285	24	$1/2^8$	$2^2, 4^2$	$2^7, 4^2$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\langle 8, 2\rangle$	1	2
286	24	$1/2^8$	$2^2, 4^2$	$2^4, 4^4$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\langle 8, 2\rangle$	2	2, 8
287	24	$1/2^8$	$2^2, 4^2$	2^{10}	D_4	$\langle 8, 3\rangle$	1	
288	24	$1/2^8$	$2^4, 4^2$	2^6	D_4	$\langle 8, 3\rangle$	1	
289	24	$1/2^8$	2^6	2^7	$\mathbb{Z}_2^3$	$\langle 8, 5\rangle$	11	$4^3, 6^2, 8^3, 12^2, 16$
290	24	$1/2^8$	2^5	2^{10}	$\mathbb{Z}_2^3$	$\langle 8, 5\rangle$	14	$0^4, 4^7, 6, 8^2$
291	24	$1/2^8$	$2^2, 6^2$	2^7	D_6	$\langle 12, 4\rangle$	1	
292	24	$1/2^8$	$2^2, 3^2, 6^2$	2^5	D_6	$\langle 12, 4\rangle$	1	
293	24	$1/2^8$	$2^3, 6$	2^{10}	D_6	$\langle 12, 4\rangle$	1	
294	24	$1/2^8$	$2^3, 3, 6$	2^6	D_6	$\langle 12, 4\rangle$	1	
295	24	$1/2^8$	$2, 4^3$	4^4	$\mathbb{Z}_4^2$	$\langle 16, 2\rangle$	1	12
296	24	$1/2^8$	$2^2, 4^2$	$2^4, 4^2$	$\mathbb{Z}_2^2 \rtimes \mathbb{Z}_4$	$\langle 16, 3\rangle$	13	8^3
297	24	$1/2^8$	$2^2, 8^2$	2^6	D_8	$\langle 16, 7\rangle$	2	
298	24	$1/2^8$	$2^2, 4^2$	$2^4, 4^2$	$\mathbb{Z}_2^2 \times \mathbb{Z}_4$	$\langle 16, 10\rangle$	10	$8^4, 12^4, 16^2$
299	24	$1/2^8$	$2^2, 4^2$	2^7	$\mathbb{Z}_2 \times D_4$	$\langle 16, 11\rangle$	7	
300	24	$1/2^8$	$2^4, 4^2$	2^5	$\mathbb{Z}_2 \times D_4$	$\langle 16, 11\rangle$	14	
301	24	$1/2^8$	$2^3, 4$	$2^4, 4^4$	$\mathbb{Z}_2 \times D_4$	$\langle 16, 11\rangle$	1	

TABLE 15. Minimal product-quotient surfaces of general type with $q = 0$, $p_g = 3$, and $K^2 \in \{26, 25, 24\}$.

No.	K_S^2	Sing(X)	t_1	t_2	G	Id	N	deg(Φ_S)
302	24	$1/2^8$	$2^3, 4^2$	$2^4, 4$	$\mathbb{Z}_2 \times D_4$	$\langle 16, 11 \rangle$	2	
303	24	$1/2^8$	$2^3, 4$	2^{10}	$\mathbb{Z}_2 \times D_4$	$\langle 16, 11 \rangle$	27	0
304	24	$1/2^8$	2^5	2^7	$\mathbb{Z}_2 \times D_4$	$\langle 16, 11 \rangle$	4	16
305	24	$1/2^8$	$2^4, 4$	2^6	$\mathbb{Z}_2 \times D_4$	$\langle 16, 11 \rangle$	14	8^2
306	24	$1/2^8$	$2^3, 4$	$2^7, 4^2$	$\mathbb{Z}_2 \times D_4$	$\langle 16, 11 \rangle$	9	
307	24	$1/2^8$	$2, 4^3$	2^6	$D_4 \rtimes \mathbb{Z}_2$	$\langle 16, 13 \rangle$	1	
308	24	$1/2^8$	2^5	2^7	$\mathbb{Z}_2^4$	$\langle 16, 14 \rangle$	13	$8^5, 12^4, 16^4$
309	24	$1/2^8$	$2^2, 3^2$	$3, 6^4$	$\mathbb{Z}_3 \times S_3$	$\langle 18, 3 \rangle$	3	0, 6
310	24	$1/2^8$	$2, 3^2, 6$	$2^2, 3^3$	$\mathbb{Z}_3 \times S_3$	$\langle 18, 3 \rangle$	2	
311	24	$1/2^8$	$2^4, 3^3$	$3, 6^2$	$\mathbb{Z}_3 \times S_3$	$\langle 18, 3 \rangle$	1	
312	24	$1/2^8$	$2, 3^4, 6$	$2^2, 3^2$	$\mathbb{Z}_3 \times S_3$	$\langle 18, 3 \rangle$	3	6
313	24	$1/2^8$	$2^2, 3^2$	$2^4, 3^3$	$\mathbb{Z}_3 \rtimes S_3$	$\langle 18, 4 \rangle$	2	
314	24	$1/2^8$	$2^2, 3^3$	$2^4, 3$	$\mathbb{Z}_3 \rtimes S_3$	$\langle 18, 4 \rangle$	2	
315	24	$1/2^8$	$2^3, 6$	$2^4, 4^2$	$\mathbb{Z}_3 \rtimes D_4$	$\langle 24, 8 \rangle$	1	
316	24	$1/2^8$	$2^2, 4^2$	$2^3, 3, 6$	$\mathbb{Z}_3 \rtimes D_4$	$\langle 24, 8 \rangle$	1	
317	24	$1/2^8$	$2^2, 3^3$	$2^2, 4^2$	S_4	$\langle 24, 12 \rangle$	1	
318	24	$1/2^8$	$2, 4^3$	$2^4, 3$	S_4	$\langle 24, 12 \rangle$	1	
319	24	$1/2^8$	2^{10}	$3, 4^2$	S_4	$\langle 24, 12 \rangle$	1	
320	24	$1/2^8$	$2^2, 3^2$	$2^4, 4^2$	S_4	$\langle 24, 12 \rangle$	1	
321	24	$1/2^8$	2^5	4^4	S_4	$\langle 24, 12 \rangle$	1	
322	24	$1/2^8$	$2^3, 3, 6$	2^5	$\mathbb{Z}_2^2 \times S_3$	$\langle 24, 14 \rangle$	3	
323	24	$1/2^8$	$2^3, 6$	2^7	$\mathbb{Z}_2^2 \times S_3$	$\langle 24, 14 \rangle$	11	
324	24	$1/2^8$	2^5	2^6	$\mathbb{Z}_2^2 \times S_3$	$\langle 24, 14 \rangle$	3	
325	24	$1/2^8$	$2^2, 14^2$	2^5	D_{14}	$\langle 28, 3 \rangle$	2	
326	24	$1/2^8$	$2^3, 14$	2^6	D_{14}	$\langle 28, 3 \rangle$	1	
327	24	$1/2^8$	$2, 4^3$	$2^2, 4^2$	$G(32, 6)$	$\langle 32, 6 \rangle$	1	
328	24	$1/2^8$	$2^2, 4^2$	$2^2, 8^2$	$D_4 \rtimes \mathbb{Z}_4$	$\langle 32, 9 \rangle$	6	
329	24	$1/2^8$	$2^3, 4^2$	$4^2, 8$	$\mathbb{Z}_4 \wr \mathbb{Z}_2$	$\langle 32, 11 \rangle$	2	
330	24	$1/2^8$	$2, 4^3$	$2^2, 4^2$	$\mathbb{Z}_4 \times D_4$	$\langle 32, 25 \rangle$	4	
331	24	$1/2^8$	$2^4, 4$	2^5	$\mathbb{Z}_2^2 \wr \mathbb{Z}_2$	$\langle 32, 27 \rangle$	3	
332	24	$1/2^8$	$2^3, 4$	$2^4, 4^2$	$\mathbb{Z}_2^2 \wr \mathbb{Z}_2$	$\langle 32, 27 \rangle$	27	
333	24	$1/2^8$	$2, 4^3$	2^5	$\mathbb{Z}_2^2 \wr \mathbb{Z}_2$	$\langle 32, 27 \rangle$	1	
334	24	$1/2^8$	$2^2, 4^2$	$2^4, 4$	$\mathbb{Z}_2^2 \wr \mathbb{Z}_2$	$\langle 32, 27 \rangle$	3	
335	24	$1/2^8$	$2^3, 4$	2^7	$\mathbb{Z}_2^2 \wr \mathbb{Z}_2$	$\langle 32, 27 \rangle$	10	
336	24	$1/2^8$	$2^3, 4$	$2^4, 4^2$	$\mathbb{Z}_2^2 \rtimes D_4$	$\langle 32, 28 \rangle$	9	
337	24	$1/2^8$	$2^2, 4^2$	$2^4, 4$	$\mathbb{Z}_2^2 \rtimes D_4$	$\langle 32, 28 \rangle$	7	
338	24	$1/2^8$	$2^2, 4^2$	$2^4, 4$	$\mathbb{Z}_4 \rtimes D_4$	$\langle 32, 34 \rangle$	3	
339	24	$1/2^8$	$2^3, 8$	2^6	$\mathbb{Z}_2 \times D_8$	$\langle 32, 39 \rangle$	5	
340	24	$1/2^8$	$2^4, 4$	2^5	$\mathbb{Z}_2 \times D_8$	$\langle 32, 39 \rangle$	1	
341	24	$1/2^8$	$2^2, 8^2$	2^5	$\mathbb{Z}_2 \times D_8$	$\langle 32, 39 \rangle$	4	
342	24	$1/2^8$	$2^3, 4^2$	$2^3, 8$	$\mathbb{Z}_8 \rtimes \mathbb{Z}_2^2$	$\langle 32, 43 \rangle$	1	
343	24	$1/2^8$	$2^2, 8^2$	2^5	$\mathbb{Z}_8 \rtimes \mathbb{Z}_2^2$	$\langle 32, 43 \rangle$	1	
344	24	$1/2^8$	$2^4, 4$	2^5	$\mathbb{Z}_2^2 \times D_4$	$\langle 32, 46 \rangle$	5	
345	24	$1/2^8$	$2^4, 4$	2^5	$Q_8 \rtimes \mathbb{Z}_2^2$	$\langle 32, 49 \rangle$	1	

TABLE 16. Minimal product-quotient surfaces of general type with $q = 0$, $p_g = 3$, and $K^2 = 24$.

No.	K_S^2	Sing(X)	t_1	t_2	G	Id	N	deg(Φ_S)
346	24	$1/2^8$	$2, 6^2$	$2^4, 3^3$	$S_3 \times S_3$	$\langle 36, 10\rangle$	2	
347	24	$1/2^8$	$2^2, 3^2, 6^2$	$2^3, 3$	$S_3 \times S_3$	$\langle 36, 10\rangle$	3	
348	24	$1/2^8$	$2^2, 3, 6$	$2^2, 6^2$	$S_3 \times S_3$	$\langle 36, 10\rangle$	1	
349	24	$1/2^8$	$2^3, 3$	$3, 6^4$	$S_3 \times S_3$	$\langle 36, 10\rangle$	2	
350	24	$1/2^8$	$2^2, 3, 6$	$2^4, 3$	$S_3 \times S_3$	$\langle 36, 10\rangle$	1	
351	24	$1/2^8$	$2^3, 3, 6$	$2^3, 6$	$S_3 \times S_3$	$\langle 36, 10\rangle$	2	
352	24	$1/2^8$	$2^2, 6^2$	6^3	$\mathbb{Z}_6 \times S_3$	$\langle 36, 12\rangle$	1	
353	24	$1/2^8$	$2^2, 3, 6$	$2^2, 6^2$	$G(36, 13)$	$\langle 36, 13\rangle$	1	
354	24	$1/2^8$	$2^2, 3, 6$	$2^4, 3$	$G(36, 13)$	$\langle 36, 13\rangle$	1	
355	24	$1/2^8$	$2^2, 8^2$	$2^3, 6$	$\mathbb{Z}_3 \rtimes D_8$	$\langle 48, 15\rangle$	2	
356	24	$1/2^8$	$2^2, 3^2$	$2^2, 8^2$	$\mathrm{GL}(2, 3)$	$\langle 48, 29\rangle$	2	
357	24	$1/2^8$	$2, 4^4$	$3, 4^2$	$\mathcal{A}_4 \rtimes \mathbb{Z}_4$	$\langle 48, 30\rangle$	1	
358	24	$1/2^8$	2^5	2^5	$\mathbb{Z}_2 \times D_{12}$	$\langle 48, 36\rangle$	1	
359	24	$1/2^8$	$2^3, 4$	2^6	$S_3 \times D_4$	$\langle 48, 38\rangle$	2	
360	24	$1/2^8$	$2^3, 6$	$2^4, 4$	$S_3 \times D_4$	$\langle 48, 38\rangle$	5	
361	24	$1/2^8$	$2^3, 3, 6$	$2^3, 4$	$S_3 \times D_4$	$\langle 48, 38\rangle$	1	
362	24	$1/2^8$	$2, 4, 6$	$2^7, 4^2$	$\mathbb{Z}_2 \times S_4$	$\langle 48, 48\rangle$	?	
363	24	$1/2^8$	$2^2, 4^2$	$2^2, 4^2$	$\mathbb{Z}_2 \times S_4$	$\langle 48, 48\rangle$	1	
364	24	$1/2^8$	$2, 4, 6$	2^{10}	$\mathbb{Z}_2 \times S_4$	$\langle 48, 48\rangle$	?	
365	24	$1/2^8$	$2^3, 6$	$2^4, 4$	$\mathbb{Z}_2 \times S_4$	$\langle 48, 48\rangle$	3	
366	24	$1/2^8$	$2, 4, 6$	$2^4, 4^4$	$\mathbb{Z}_2 \times S_4$	$\langle 48, 48\rangle$	3	
367	24	$1/2^8$	$2^3, 3$	$2^4, 4^2$	$\mathbb{Z}_2 \times S_4$	$\langle 48, 48\rangle$	7	
368	24	$1/2^8$	$2^2, 3, 6$	$2^2, 4^2$	$\mathbb{Z}_2 \times S_4$	$\langle 48, 48\rangle$	2	
369	24	$1/2^8$	$2, 4^4$	$2^3, 3$	$\mathbb{Z}_2 \times S_4$	$\langle 48, 48\rangle$	1	
370	24	$1/2^8$	$2^3, 3, 6$	$2^3, 4$	$\mathbb{Z}_2 \times S_4$	$\langle 48, 48\rangle$	2	
371	24	$1/2^8$	$2^2, 4^2$	2^5	$\mathbb{Z}_2 \times S_4$	$\langle 48, 48\rangle$	1	
372	24	$1/2^8$	$2, 4^3$	$2^3, 6$	$\mathbb{Z}_2 \times S_4$	$\langle 48, 48\rangle$	1	
373	24	$1/2^8$	2^5	2^5	$\mathbb{Z}_2^3 \times S_3$	$\langle 48, 51\rangle$	1	
374	24	$1/2^8$	$2, 3^2, 6$	$2^2, 3^2$	$\mathrm{He3} \rtimes \mathbb{Z}_2$	$\langle 54, 5\rangle$	1	
375	24	$1/2^8$	$2, 3^2, 6$	$2^2, 3^2$	$\mathbb{Z}_3^2 \rtimes S_3$	$\langle 54, 8\rangle$	1	
376	24	$1/2^8$	$2, 3^2, 6$	$3, 6^2$	$S_3 \times \mathbb{Z}_3^2$	$\langle 54, 12\rangle$	9	$12, (16, 18), (13, 15), 18, 24$
377	24	$1/2^8$	$2, 3^2, 6$	$2^2, 3^2$	$G(54, 13)$	$\langle 54, 13\rangle$	4	
378	24	$1/2^8$	$2^2, 4^2$	$2^3, 14$	$D_{14} \rtimes \mathbb{Z}_2$	$\langle 56, 7\rangle$	1	
379	24	$1/2^8$	$2^3, 14$	2^5	$\mathbb{Z}_2^2 \times D_7$	$\langle 56, 12\rangle$	3	
380	24	$1/2^8$	$2, 5^2$	$2^4, 3^3$	$\mathcal{A}_5$	$\langle 60, 5\rangle$	1	
381	24	$1/2^8$	$2^2, 3^2$	$2^2, 5^2$	$\mathcal{A}_5$	$\langle 60, 5\rangle$	2	
382	24	$1/2^8$	$2^2, 4^2$	$4^2, 8$	$G(64, 8)$	$\langle 64, 8\rangle$	1	
383	24	$1/2^8$	$2, 4, 8$	$2, 4^4$	$G(64, 8)$	$\langle 64, 8\rangle$	4	
384	24	$1/2^8$	$2, 4^3$	4^3	$G(64, 23)$	$\langle 64, 23\rangle$	6	
385	24	$1/2^8$	$2^3, 4$	$2^4, 4$	$G(64, 73)$	$\langle 64, 73\rangle$	2	
386	24	$1/2^8$	$2^3, 4$	$2^4, 4$	$G(64, 128)$	$\langle 64, 128\rangle$	1	
387	24	$1/2^8$	$2^3, 8$	2^5	$G(64, 128)$	$\langle 64, 128\rangle$	1	
388	24	$1/2^8$	$2^2, 8^2$	$2^3, 4$	$G(64, 128)$	$\langle 64, 128\rangle$	1	

TABLE 17. Minimal product-quotient surfaces of general type with $q = 0$, $p_g = 3$, and $K^2 = 24$.

No.	K_S^2	Sing(X)	t_1	t_2	G	Id	N	deg(Φ_S)
389	24	$1/2^8$	$2^2,4^2$	$2^3,8$	$G(64,130)$	$\langle 64,130\rangle$	1	
390	24	$1/2^8$	$2^2,4^2$	$2^3,8$	$D_4 \rtimes D_4$	$\langle 64,134\rangle$	1	
391	24	$1/2^8$	$2,4^3$	$2^3,4$	$D_4 \rtimes D_4$	$\langle 64,134\rangle$	1	
392	24	$1/2^8$	$2^3,4$	$2^4,4$	$\mathbb{Z}_2 \wr \mathbb{Z}_2^2$	$\langle 64,138\rangle$	5	
393	24	$1/2^8$	$2,4^3$	$2^3,4$	$\mathbb{Z}_2 \wr \mathbb{Z}_2^2$	$\langle 64,138\rangle$	2	
394	24	$1/2^8$	$2^3,4$	$2^4,4$	$\mathbb{Z}_4 \rtimes D_8$	$\langle 64,140\rangle$	1	
395	24	$1/2^8$	$2^2,8^2$	$2^3,4$	$\mathbb{Z}_4 \rtimes D_8$	$\langle 64,140\rangle$	1	
396	24	$1/2^8$	$2^2,4^2$	$2^3,8$	$\mathbb{Z}_2^2 \rtimes D_8$	$\langle 64,147\rangle$	1	
397	24	$1/2^8$	$2^2,4^2$	$2^3,8$	$G(64,150)$	$\langle 64,150\rangle$	1	
398	24	$1/2^8$	$2,6,12$	$2^4,3$	$\mathbb{Z}_3^2 \rtimes D_4$	$\langle 72,23\rangle$	1	
399	24	$1/2^8$	$2,4,6$	$2^2,3^2,6^2$	$S_3 \wr \mathbb{Z}_2$	$\langle 72,40\rangle$	1	
400	24	$1/2^8$	$2^2,3,6$	$2^2,3^2$	$\mathbb{Z}_3 \rtimes S_4$	$\langle 72,43\rangle$	1	
401	24	$1/2^8$	$2^2,3^2$	$2^2,4^2$	$\mathbb{Z}_3 \rtimes S_4$	$\langle 72,43\rangle$	1	
402	24	$1/2^8$	$2^2,3^2$	6^3	$S_3 \times \mathcal{A}_4$	$\langle 72,44\rangle$	1	
403	24	$1/2^8$	$2^3,6$	2^5	$\mathbb{Z}_2 \times S_3 \times S_3$	$\langle 72,46\rangle$	1	
404	24	$1/2^8$	$2^3,6$	$2^3,14$	$S_3 \times D_7$	$\langle 84,8\rangle$	1	
405	24	$1/2^8$	$2^3,4$	2^5	$\mathbb{Z}_2^2 \rtimes D_{12}$	$\langle 96,89\rangle$	1	
406	24	$1/2^8$	$2^3,6$	$2^3,8$	$S_3 \times D_8$	$\langle 96,117\rangle$	1	
407	24	$1/2^8$	$2,4,12$	$2,4^3$	$\mathbb{Z}_4 \times S_4$	$\langle 96,186\rangle$	3	
408	24	$1/2^8$	$2,4,12$	$2^4,4$	$\mathbb{Z}_4 \rtimes S_4$	$\langle 96,187\rangle$	2	
409	24	$1/2^8$	$2^2,8^2$	$2^3,3$	$G(96,193)$	$\langle 96,193\rangle$	2	
410	24	$1/2^8$	$2,4,6$	$2^4,4^2$	$\mathrm{GL}(2,\mathbb{Z}_4)$	$\langle 96,195\rangle$	6	
411	24	$1/2^8$	$2,4,6$	$2,4^4$	$\mathrm{GL}(2,\mathbb{Z}_4)$	$\langle 96,195\rangle$	3	
412	24	$1/2^8$	$2^4,4$	$3,4^2$	$\mathbb{Z}_2^2 \rtimes S_4$	$\langle 96,227\rangle$	1	
413	24	$1/2^8$	$2,4^3$	$2^3,3$	$\mathbb{Z}_2^2 \rtimes S_4$	$\langle 96,227\rangle$	2	
414	24	$1/2^8$	$2,6^2$	$2^4,3$	$G(108,17)$	$\langle 108,17\rangle$	1	
415	24	$1/2^8$	$2^3,4$	$2^3,14$	$D_4 \times D_7$	$\langle 112,31\rangle$	1	
416	24	$1/2^8$	$2,4,5$	$3,6^4$	S_5	$\langle 120,34\rangle$	2	
417	24	$1/2^8$	$2^3,5$	$3,6^2$	S_5	$\langle 120,34\rangle$	1	
418	24	$1/2^8$	$2,4^3$	$2,5,6$	S_5	$\langle 120,34\rangle$	1	
419	24	$1/2^8$	$2,6^2$	$2^2,5^2$	S_5	$\langle 120,34\rangle$	1	
420	24	$1/2^8$	$2^3,6$	$4^2,5$	S_5	$\langle 120,34\rangle$	1	
421	24	$1/2^8$	$2,3^4,6$	$2,4,5$	S_5	$\langle 120,34\rangle$	1	
422	24	$1/2^8$	$2,4,5$	$2^2,3^2,6^2$	S_5	$\langle 120,34\rangle$	1	
423	24	$1/2^8$	$2,5,6$	$2^4,4$	S_5	$\langle 120,34\rangle$	1	
424	24	$1/2^8$	$2,5,10$	$2^2,3,6$	$\mathbb{Z}_2 \times \mathcal{A}_5$	$\langle 120,35\rangle$	1	
425	24	$1/2^8$	$2,5,10$	2^5	$\mathbb{Z}_2 \times \mathcal{A}_5$	$\langle 120,35\rangle$	1	
426	24	$1/2^8$	$2,10^2$	$2^3,6$	$\mathbb{Z}_2 \times \mathcal{A}_5$	$\langle 120,35\rangle$	1	
427	24	$1/2^8$	$2,3,10$	2^7	$\mathbb{Z}_2 \times \mathcal{A}_5$	$\langle 120,35\rangle$	1	
428	24	$1/2^8$	$2^2,5^2$	$2^3,3$	$\mathbb{Z}_2 \times \mathcal{A}_5$	$\langle 120,35\rangle$	1	
429	24	$1/2^8$	$2,4,8$	$2,4^3$	$G(128,75)$	$\langle 128,75\rangle$	4	
430	24	$1/2^8$	$2^3,4$	$2^3,8$	$G(128,327)$	$\langle 128,327\rangle$	1	
431	24	$1/2^8$	$2^3,4$	$2^3,8$	$G(128,928)$	$\langle 128,928\rangle$	1	
432	24	$1/2^8$	$2,4,5$	$2,4^4$	$\mathbb{Z}_2^4 \rtimes D_5$	$\langle 160,234\rangle$	5	
433	24	$1/2^8$	$2,4,5$	$2^4,4^2$	$\mathbb{Z}_2^4 \rtimes D_5$	$\langle 160,234\rangle$	?	

TABLE 18. Minimal product-quotient surfaces of general type with $q = 0$, $p_g = 3$, and $K^2 = 24$.

No.	K_S^2	Sing(X)	t_1	t_2	G	Id	N	$\deg(\Phi_S)$
434	24	$1/2^8$	$2,6,9$	$3,6^2$	$\mathbb{Z}_3 \wr S_3$	$\langle 162, 10\rangle$	4	
435	24	$1/2^8$	$2,7^2$	$2^2,3^2$	PSL$(2,7)$	$\langle 168, 42\rangle$	1	
436	24	$1/2^8$	$2^4,7$	$3^2,4$	PSL$(2,7)$	$\langle 168, 42\rangle$	2	
437	24	$1/2^8$	$2,3,8$	$2,4^4$	$G(192,181)$	$\langle 192, 181\rangle$	1	
438	24	$1/2^8$	$2,4,6$	$2^4,4$	$G(192,955)$	$\langle 192, 955\rangle$	5	
439	24	$1/2^8$	$2,4,6$	$2,4^3$	$G(192,955)$	$\langle 192, 955\rangle$	1	
440	24	$1/2^8$	$2,4,6$	$2^2,6^2$	$G(216,87)$	$\langle 216, 87\rangle$	1	
441	24	$1/2^8$	$2,4,10$	$2^3,6$	$\mathbb{Z}_2 \times S_5$	$\langle 240, 189\rangle$	1	
442	24	$1/2^8$	$2,10^2$	$2^3,3$	$\mathbb{Z}_2^2 \times \mathcal{A}_5$	$\langle 240, 190\rangle$	1	
443	24	$1/2^8$	$2,4,5$	$2^2,8^2$	$G(320,1582)$	$\langle 320, 1582\rangle$	5	
444	24	$1/2^8$	$2,4,10$	$2^3,4$	$G(320,1636)$	$\langle 320, 1636\rangle$	2	
445	24	$1/2^8$	$2,5^2$	$2^2,3^2$	$\mathcal{A}_6$	$\langle 360, 118\rangle$	1	
446	24	$1/2^8$	$2,3,10$	$2^2,3,6$	$S_3 \times \mathcal{A}_5$	$\langle 360, 121\rangle$	1	
447	24	$1/2^8$	$2,4,6$	$2^3,8$	$G(384,5602)$	$\langle 384, 5602\rangle$	3	
448	24	$1/2^8$	$2,3,8$	$2,3^2,6$	AGL$(2,3)$	$\langle 432, 734\rangle$	2	
449	24	$1/2^8$	$2,3,8$	$2,6,21$	$G(1008,881)$	$\langle 1008, 881\rangle$	4	
450	24	$2/5^2, 1/2^4$	$2,4,5$	$2^4,4,5$	$\mathbb{Z}_2^4 \rtimes D_5$	$\langle 160, 234\rangle$	?	
451	24	$2/5^2, 1/2^4$	$2,4,5$	$2,4^3,5$	$\mathbb{Z}_2^4 \rtimes D_5$	$\langle 160, 234\rangle$	4	
452	24	$2/5^2, 1/2^4$	$2,4,5$	$2^2,8,10$	$G(320,1582)$	$\langle 320, 1582\rangle$	4	
453	24	$2/5^4$	$2^4,5^2$	$3^2,5$	$\mathcal{A}_5$	$\langle 60, 5\rangle$	1	
454	24	$2/5^4$	$2^4,5$	$3,5^2$	$\mathcal{A}_5$	$\langle 60, 5\rangle$	1	
455	24	$2/5^4$	$2,5^2$	$2^4,5^2$	$\mathbb{Z}_2^4 \rtimes \mathbb{Z}_5$	$\langle 80, 49\rangle$	5	
456	24	$2/5^4$	$3,15^2$	$3^2,5$	$\mathbb{Z}_3 \times \mathcal{A}_5$	$\langle 180, 19\rangle$	1	
457	24	$2/5^4$	$2,4,5$	$2,4^2,5^2$	$\mathbb{Z}_2^4 \rtimes D_5$	$\langle 160, 234\rangle$	6	
458	24	$2/5^4$	$2,5^2$	$2,5^2$	$G(1280,\cdot)$	$\langle 1280, 1116310\rangle$	2	
459	24	$1/4^2, 3/4^2$	$2^3,4$	$2^9,4$	$\mathbb{Z}_2 \times D_4$	$\langle 16, 11\rangle$	6	0
460	24	$1/4^2, 3/4^2$	$2^9,4$	$3,4^2$	S_4	$\langle 24, 12\rangle$	1	
461	24	$1/4^2, 3/4^2$	$2^3,4$	$3^4,4^2$	S_4	$\langle 24, 12\rangle$	2	
462	24	$1/4^2, 3/4^2$	$3,4^2$	$3^4,4^2$	$G(36,9)$	$\langle 36, 9\rangle$	1	
463	24	$1/4^2, 3/4^2$	$2,4,6$	$2^9,4$	$\mathbb{Z}_2 \times S_4$	$\langle 48, 48\rangle$	?	
464	24	$1/4^2, 3/4^2$	$2,4,5$	$3^4,4^2$	S_5	$\langle 120, 34\rangle$	2	
465	24	$1/4^2, 3/4^2$	$3,4^2$	$4,7^2$	PSL$(2,7)$	$\langle 168, 42\rangle$	2	
466	24	$1/4^2, 3/4^2$	$2,4,6$	$2^2,4,10$	$\mathbb{Z}_2 \times S_5$	$\langle 240, 189\rangle$	1	
467	24	$1/4, 1/2^4, 3/4$	$2,3^3,4,6$	$2,4,6$	$S_3 \wr \mathbb{Z}_2$	$\langle 72, 40\rangle$	1	
468	24	$1/4, 1/2^4, 3/4$	$2,3^3,4,6$	$2,4,5$	S_5	$\langle 120, 34\rangle$	1	
469	24	$1/4, 1/2^4, 3/4$	$2,4,14$	$2,4,14$	$D_7 \wr \mathbb{Z}_2$	$\langle 392, 37\rangle$	2	
470	24	$1/3^2, 1/2^2, 2/3^2$	$2,4,6$	$2^6,3^2,4$	$\mathbb{Z}_2 \times S_4$	$\langle 48, 48\rangle$	?	
471	24	$1/3^2, 1/2^2, 2/3^2$	$2,3,7^2$	$3^2,4$	PSL$(2,7)$	$\langle 168, 42\rangle$	4	
472	24	$3/10^2, 1/2^2$	$2,3,10$	$2,8,10$	$G(720,764)$	$\langle 720, 764\rangle$	2	
473	24	$3/10^2, 1/2^2$	$2,3,10$	$2,4,10$	$G(1320,133)$	$\langle 1320, 133\rangle$	2	
474	24	$3/8^2, 1/2, 3/4$	$2,3,8$	$2,4^3,8$	$G(192,181)$	$\langle 192, 181\rangle$	2	
475	23	$1/3^3, 2/3^3$	3^4	3^6	$\mathbb{Z}_3^2$	$\langle 9, 2\rangle$	6	$6^5,9$
476	23	$1/3^3, 2/3^3$	$2^6,3^3$	$3,4^2$	S_4	$\langle 24, 12\rangle$	1	
477	23	$1/3^3, 2/3^3$	$2^2,3^2$	$2^4,3,6$	$\mathbb{Z}_2 \times \mathcal{A}_4$	$\langle 24, 13\rangle$	2	8
478	23	$1/3^3, 2/3^3$	$2^2,3^2$	$2^2,6^3$	$\mathbb{Z}_2 \times \mathcal{A}_4$	$\langle 24, 13\rangle$	1	

TABLE 19. Minimal product-quotient surfaces of general type with $q = 0$, $p_g = 3$, and $K^2 \in \{24, 23\}$.

No.	K_S^2	Sing(X)	t_1	t_2	G	Id	N	deg(Φ_S)
479	23	$1/3^3, 2/3^3$	$2, 6^2$	$2^6, 3^3$	$\mathbb{Z}_2 \times \mathcal{A}_4$	$\langle 24, 13\rangle$	1	
480	23	$1/3^3, 2/3^3$	3^4	3^4	He3	$\langle 27, 3\rangle$	5	
481	23	$1/3^3, 2/3^3$	$2, 3^3$	3^4	$\mathbb{Z}_3 \times \mathcal{A}_4$	$\langle 36, 11\rangle$	4	
482	23	$1/3^3, 2/3^3$	$3^2, 6$	3^6	$\mathbb{Z}_3 \times \mathcal{A}_4$	$\langle 36, 11\rangle$	6	
483	23	$1/3^3, 2/3^3$	$2, 3, 4^2, 6$	$3, 4^2$	$\mathcal{A}_4 \rtimes \mathbb{Z}_4$	$\langle 48, 30\rangle$	2	
484	23	$1/3^3, 2/3^3$	$2, 3, 4^2, 6$	$2^3, 3$	$\mathbb{Z}_2 \times S_4$	$\langle 48, 48\rangle$	3	
485	23	$1/3^3, 2/3^3$	$2, 4, 6$	$2^6, 3^3$	$\mathbb{Z}_2 \times S_4$	$\langle 48, 48\rangle$	?	
486	23	$1/3^3, 2/3^3$	$2, 6^2$	$2^4, 3, 6$	$\mathbb{Z}_2^2 \times \mathcal{A}_4$	$\langle 48, 49\rangle$	6	8
487	23	$1/3^3, 2/3^3$	$2, 6^2$	$2^2, 6^3$	$\mathbb{Z}_2^2 \times \mathcal{A}_4$	$\langle 48, 49\rangle$	1	
488	23	$1/3^3, 2/3^3$	$3^2, 21$	3^4	$(\mathbb{Z}_3 \times \mathbb{Z}_7) \rtimes \mathbb{Z}_3$	$\langle 63, 3\rangle$	8	
489	23	$1/3^3, 2/3^3$	$2^2, 3^2$	$3, 12^2$	$\mathbb{Z}_3 \times S_4$	$\langle 72, 42\rangle$	1	
490	23	$1/3^3, 2/3^3$	$2^2, 3^2$	6^3	$S_3 \times \mathcal{A}_4$	$\langle 72, 44\rangle$	1	
491	23	$1/3^3, 2/3^3$	$2^2, 3, 6$	$2^2, 3^2$	$S_3 \times \mathcal{A}_4$	$\langle 72, 44\rangle$	3	
492	23	$1/3^3, 2/3^3$	$3^2, 9$	3^4	$\mathbb{Z}_3 \wr \mathbb{Z}_3$	$\langle 81, 7\rangle$	4	
493	23	$1/3^3, 2/3^3$	$3^2, 9$	3^4	He3 $\rtimes \mathbb{Z}_3$	$\langle 81, 9\rangle$	8	
494	23	$1/3^3, 2/3^3$	$2^2, 6^3$	$3^2, 4$	$G(96, 3)$	$\langle 96, 3\rangle$	3	
495	23	$1/3^3, 2/3^3$	$2, 3, 4^2, 6$	$2, 4, 6$	$\mathrm{GL}(2, \mathbb{Z}_4)$	$\langle 96, 195\rangle$	1	
496	23	$1/3^3, 2/3^3$	$3^2, 6$	3^4	$\mathbb{Z}_6^2 \rtimes \mathbb{Z}_3$	$\langle 108, 22\rangle$	12	
497	23	$1/3^3, 2/3^3$	$2, 3^3$	$3^2, 6$	$\mathcal{A}_4 \times \mathcal{A}_4$	$\langle 144, 184\rangle$	2	
498	23	$1/3^3, 2/3^3$	$2, 6^2$	$2^2, 3, 6$	$\mathbb{Z}_2 \times S_3 \times \mathcal{A}_4$	$\langle 144, 190\rangle$	2	
499	23	$1/3^3, 2/3^3$	$2^2, 3, 6$	$3^2, 5$	$\mathbb{Z}_3 \times \mathcal{A}_5$	$\langle 180, 19\rangle$	1	
500	23	$1/3^3, 2/3^3$	$2, 3, 15$	3^4	$\mathbb{Z}_3 \times \mathcal{A}_5$	$\langle 180, 19\rangle$	1	
501	23	$1/3^3, 2/3^3$	$2, 3, 4^2, 6$	$2, 3, 8$	$G(192, 181)$	$\langle 192, 181\rangle$	2	
502	23	$1/3^3, 2/3^3$	$3^2, 4$	3^4	$\mathrm{ASL}(2, 3)$	$\langle 216, 153\rangle$	2	
503	23	$1/3^3, 2/3^3$	$3^2, 9$	$3^2, 9$	(He3 $\rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_3$	$\langle 243, 26\rangle$	7	
504	23	$1/3^3, 2/3^3$	$3^2, 9$	$3^2, 9$	$G(243, 28)$	$\langle 243, 28\rangle$	18	
505	23	$1/3^3, 2/3^3$	$3^2, 6$	$3^2, 21$	$(\mathcal{A}_4 \times \mathbb{Z}_7) \rtimes \mathbb{Z}_3$	$\langle 252, 27\rangle$	4	
506	23	$1/3^3, 2/3^3$	$2, 3, 12$	$2^2, 3, 6$	$\mathcal{A}_4 \times S_4$	$\langle 288, 1024\rangle$	2	
507	23	$1/3^3, 2/3^3$	$3^2, 6$	$3^2, 9$	$(\mathbb{Z}_2^2 \times \mathrm{He3}) \rtimes \mathbb{Z}_3$	$\langle 324, 54\rangle$	9	
508	23	$1/3^3, 2/3^3$	$3^2, 6$	$3^2, 6$	$(\mathbb{Z}_3 \times \mathcal{A}_4) \rtimes \mathcal{A}_4$	$\langle 432, 526\rangle$	6	
509	23	$1/3^3, 2/3^3$	$3^2, 4$	$3^2, 21$	$\mathbb{Z}_3 \times \mathrm{PSL}(2, 7)$	$\langle 504, 157\rangle$	4	
510	23	$1/3^3, 2/3^3$	$3^2, 4$	$3^2, 6$	$G(864, 2666)$	$\langle 864, 2666\rangle$	8	
511	23	$1/3^3, 2/3^3$	$2, 3, 7$	$3, 6, 8$	$G(1344, 814)$	$\langle 1344, 814\rangle$	16	
512	23	$3/8, 1/2^4, 5/8$	$2^3, 16$	$2^3, 16$	$\mathbb{Z}_2 \times D_{16}$	$\langle 64, 186\rangle$	2	
513	23	$1/3, 1/2^6, 2/3$	$2^2, 4, 6$	$2^3, 12$	$\mathbb{Z}_2 \times D_{12}$	$\langle 48, 36\rangle$	1	
514	23	$1/3, 1/2^6, 2/3$	$2^2, 4, 6$	$2^3, 12$	$S_3 \times D_4$	$\langle 48, 38\rangle$	1	
515	23	$1/3, 1/2^6, 2/3$	$2^2, 3, 4$	$2^2, 4, 6$	$\mathbb{Z}_2 \times S_4$	$\langle 48, 48\rangle$	2	
516	23	$1/3, 1/2^6, 2/3$	$2, 4, 6$	$2^7, 3, 4$	$\mathbb{Z}_2 \times S_4$	$\langle 48, 48\rangle$	?	
517	23	$1/3, 1/2^6, 2/3$	$2, 4, 6$	$2^4, 3, 4^3$	$\mathbb{Z}_2 \times S_4$	$\langle 48, 48\rangle$	?	
518	23	$1/3, 1/2^6, 2/3$	$2^2, 3, 4$	$3, 4, 8$	$G(96, 64)$	$\langle 96, 64\rangle$	1	
519	23	$1/3, 1/2^6, 2/3$	$2, 6, 8$	$2^2, 4, 6$	$\mathbb{Z}_2 \times \mathrm{GL}(2, 3)$	$\langle 96, 189\rangle$	1	
520	23	$1/3, 1/2^6, 2/3$	$2, 6, 18$	$2^3, 9$	$S_3 \times D_9$	$\langle 108, 16\rangle$	3	
521	23	$1/3, 1/2^6, 2/3$	$2, 4, 6$	$2^2, 4, 6$	$\mathbb{Z}_2 \times S_5$	$\langle 240, 189\rangle$	3	
522	23	$1/3, 1/2^6, 2/3$	$2, 4, 6$	$2, 6, 8$	$\mathbb{Z}_2 \times \mathrm{SO}(3, 7)$	$\langle 672, 1254\rangle$	4	

TABLE 20. Minimal product-quotient surfaces of general type with $q = 0$, $p_g = 3$, and $K^2 = 23$.

No.	K_S^2	Sing(X)	t_1	t_2	G	Id	N	deg(Φ_S)
523	25	$1/7, 2/7^2$	$2, 4, 7$	$3^3, 7$	PSL(2, 7)	$\langle 168, 42\rangle$	2	
524	25	$1/7, 2/7^2$	$2, 4, 7$	$3, 6, 14$	$\mathbb{Z}_2 \times$ PSL(2, 7)	$\langle 336, 209\rangle$	1	
525	25	$1/7, 2/7^2$	$2, 3, 7$	$4, 7, 8$	$G(1344, 814)$	$\langle 1344, 814\rangle$	8	
526	24	$1/5, 1/3, 2/3, 4/5$	$2, 6, 10$	$2^2, 3, 5$	$\mathbb{Z}_2 \times \mathcal{A}_5$	$\langle 120, 35\rangle$	1	
527	24	$1/6^2, 1/2^2, 2/3$	$2, 4, 6$	$4^4, 6$	GL$(2, \mathbb{Z}_4)$	$\langle 96, 195\rangle$	2	
528	24	$1/6^2, 1/2^2, 2/3$	$2, 4, 6$	$2^3, 4^2, 6$	GL$(2, \mathbb{Z}_4)$	$\langle 96, 195\rangle$	14	
529	24	$1/6^2, 1/2^2, 2/3$	$2, 5, 6$	$4, 6, 8$	SL$(2, 5) \rtimes \mathbb{Z}_2$	$\langle 240, 90\rangle$	2	
530	24	$1/6^2, 1/2^2, 2/3$	$2, 4, 6$	$4, 6, 8$	$G(384, 5604)$	$\langle 384, 5604\rangle$	4	
531	24	$1/6^2, 1/2^2, 2/3$	$2, 4, 6$	$4, 6, 8$	$G(384, 5677)$	$\langle 384, 5677\rangle$	4	
532	24	$1/4^4, 1/2^2$	$2, 4^2, 8$	$2^2, 4^2$	$\mathbb{Z}_4 \wr \mathbb{Z}_2$	$\langle 32, 11\rangle$	1	
533	24	$1/4^4, 1/2^2$	$2^3, 4$	$2^3, 4^3$	$\mathbb{Z}_2^2 \rtimes D_4$	$\langle 32, 28\rangle$	4	
534	24	$1/4^4, 1/2^2$	$2, 4, 8$	$2^3, 4^3$	$\mathbb{Z}_2 \wr \mathbb{Z}_4$	$\langle 64, 32\rangle$	4	
535	24	$1/4^4, 1/2^2$	$2, 4, 8$	$2, 4^2, 8$	$G(128, 136)$	$\langle 128, 136\rangle$	1	
536	24	$1/4^4, 1/2^2$	$2, 3, 8$	4^5	$G(192, 181)$	$\langle 192, 181\rangle$	1	
537	24	$1/8^2, 1/4, 1/2$	$2, 3, 8$	$2^2, 4^3, 8$	$G(192, 181)$	$\langle 192, 181\rangle$	3	
538	24	$1/6, 1/2^2, 5/6$	$2, 4, 6$	$2^9, 6$	$\mathbb{Z}_2 \times S_4$	$\langle 48, 48\rangle$	?	
539	24	$1/6, 1/2^2, 5/6$	$2, 4^2, 6$	$2^3, 6$	$\mathbb{Z}_2 \times S_4$	$\langle 48, 48\rangle$	1	
540	24	$1/6, 1/2^2, 5/6$	$2^4, 6$	$4^2, 6$	$\mathbb{Z}_2 \times S_4$	$\langle 48, 48\rangle$	2	
541	24	$1/6, 1/2^2, 5/6$	$2, 4, 6$	$2, 4^2, 6$	$G(192, 955)$	$\langle 192, 955\rangle$	4	
542	24	$1/6, 1/2^2, 5/6$	$2, 6, 8$	$2^3, 6$	$G(192, 956)$	$\langle 192, 956\rangle$	1	
543	24	$1/6, 1/2^2, 5/6$	$2, 6, 7$	$2, 6, 8$	SO(3, 7)	$\langle 336, 208\rangle$	2	
544	24	$1/6, 1/2^2, 5/6$	$2, 4, 6$	$2, 6, 8$	$G(768, 1086051)$	$\langle 768, 1086051\rangle$	2	
545	24	$1/4, 1/2, 5/8^2$	$2, 3, 8$	$2, 4^3, 8$	$G(192, 181)$	$\langle 192, 181\rangle$	2	
546	23	$1/5^5$	$2, 5^2$	$5^2, 15$	$\mathbb{Z}_5 \times \mathcal{A}_5$	$\langle 300, 22\rangle$	2	
547	23	$1/5, 2/5^2, 4/5$	$2^4, 5^2$	$3^2, 5$	$\mathcal{A}_5$	$\langle 60, 5\rangle$	1	
548	23	$1/5, 2/5^2, 4/5$	$2^4, 5$	$3, 5^2$	$\mathcal{A}_5$	$\langle 60, 5\rangle$	1	
549	23	$1/5, 2/5^2, 4/5$	$2, 5^2$	$3^5, 5$	$\mathcal{A}_5$	$\langle 60, 5\rangle$	2	
550	23	$1/5, 2/5^2, 4/5$	$2, 4^2, 5$	$2, 5, 6$	S_5	$\langle 120, 34\rangle$	1	
551	23	$1/5, 2/5^2, 4/5$	$2, 4, 5$	$2, 4^2, 5^2$	$\mathbb{Z}_2^4 \rtimes D_5$	$\langle 160, 234\rangle$	6	
552	23	$1/5, 2/5^2, 4/5$	$3, 15^2$	$3^2, 5$	$\mathbb{Z}_3 \times \mathcal{A}_5$	$\langle 180, 19\rangle$	1	
553	23	$1/5, 1/2^4, 4/5$	$2, 4, 5$	$2^4, 4, 5$	$\mathbb{Z}_2^4 \rtimes D_5$	$\langle 160, 234\rangle$	?	
554	23	$1/5, 1/2^4, 4/5$	$2, 4, 5$	$2, 4^3, 5$	$\mathbb{Z}_2^4 \rtimes D_5$	$\langle 160, 234\rangle$	4	
555	23	$1/5, 1/2^4, 4/5$	$2, 4, 5$	$2^2, 8, 10$	$G(320, 1582)$	$\langle 320, 1582\rangle$	4	

TABLE 21. Remaining product-quotient surfaces of general type with $q = 0$, $p_g = 3$, and $K^2 \in \{23, \ldots, 32\}$ whose minimality is not established.

ACKNOWLEDGMENTS. – The author expresses gratitude to Roberto Pignatelli for engaging discussions on the classification of product-quotient surfaces with low fixed invariants.

FUNDING. – The author acknowledges support from the project "Classificazione di superfici di genere basso" in collaboration with the University of Trento. The author is partially supported by the INdAM – GNSAGA Project, "Classification Problems in Algebraic Geometry: Lefschetz Properties and Moduli Spaces" (CUP E55F22000270001).

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Received 11 May 2024,
and in revised form 2 February 2025

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